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THE BOUNDARY LAYER NATURE OF SHOCK TRANSITION IN A REAL FLUID*

BY

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Foreword. The usual way to deal with the so-called shoch phenomenon in compressible fluids is the following. On the one hand there is the fact that in many cases of observable flow there exist narrow zones across which pressure, density and velocity undergo rapid changes. On the other hand, it is well-known that the differential equations of inviscid perfect fluids fail to supply solutions satisfying certain boundary conditions that can be realized physically. One therefore makes the assumption that these differential equations are valid in regions of the (x, y, z, t) space which are separated from each other by discontinuity surfaces whose shape is a priori unknown. From physically plausible hypotheses one then derives *necessary* conditions for the values assumed by the physical variables on either side of the discontinuity surfaces. Such conditions were first given by Riemann, and later modified by Rankine and Hugoniot. It is finally assumed—and confirmed at least in special cases—that the differential equations combined with these transition conditions are sufficient to determine both the discontinuity surfaces themselves and the continuous flows in the regions between them (see [4], pp. 116-118, 134-138).

A different approach, as suggested by R. von Mises to the author, is followed in the present paper. The sole basis is formed by the system of partial differential equations (Navier-Stokes equations) which govern the motion of a viscous, heat-conducting, compressible fluid. No additional assumptions of any kind are introduced. It is *proved* that the integrals of these equations include a class of solutions of the boundary layer type; that is to say, solutions which asymptotically converge (with vanishing viscosity) towards flow patterns entirely different from those which are obtained when from the start viscosity is neglected. For a small viscosity coefficient μ, these flows have rapid changes of the physical variables across certain narrow regions, the widths of which converge to zero as $\mu \to 0$. In the limit the values of the variables on the two sides of the transition are subject to equations which are identical with the Rankine-Hugoniot conditions. These conditions, obtained here without any hypotheses, are thus proved to be not only necessary but also *sufficient* for the existence of shocks.

1. Introduction. In a recent paper [6], R. von Mises has discussed the occurrence of a shock in the one-dimensional steady flow of a real fluid. Here "real" will be used to denote "heat conducting, viscous and compressible." If we write his fundamental equa-

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tions (1) through (5) in a dimension-free form by referring ρ, u, x, p to standard values ρ_0 , u_0 , d, $\rho_0 u_0^2$, we obtain:

$$\rho u = m, \tag{1.1}$$

$$mu + p - \sigma = C_1 m, \tag{1.2}$$

$$m\left(\frac{u^2}{2} + \frac{T}{\kappa - 1}\right) + u(p - \sigma) - H = C_2 m, \tag{1.3}$$

$$p = \rho T, \tag{1.4}$$

with

$$\sigma = \frac{4}{3R}\frac{du}{dx}, \qquad H = \frac{\kappa}{\kappa - 1} \cdot \frac{1}{PR}\frac{dT}{dx}, \tag{1.5}$$

where T is referred to $u_0^2/\overline{R}$ ($\overline{R}$ is the gR of [6]), and we use the same symbols to denote the non-dimensional quantities, but change the γ of Reference 6 to κ for convenience. Here R is the Reynolds number,

$$R = \frac{\rho_0 u_0 d}{\mu}, \tag{1.6}$$

and P the Prandtl number,

$$P = \frac{\kappa}{\kappa - 1}\overline{R}\frac{\mu}{k}. \tag{1.7}$$

These equations are based on the assumption that the fluid is a perfect gas (1.4) with constant specific heats. Further, von Mises considers that P, μ, k may vary. For convenience we shall also assume μ, k and therefore P are constant, though the results are equally true without this restriction. Finally, if we write

$$x = \frac{s}{R}, \tag{1.8}$$

the equations (1.5) become

$$\sigma = \frac{4}{3}\frac{du}{ds}, \qquad H = \frac{\kappa}{\kappa - 1}\frac{1}{P}\frac{dT}{ds}, \tag{1.5'}$$

and the set of equations (1.1) through (1.4) and (1.5′) is now independent of R.

In order to discuss the boundary layer nature of shock transition, we recall the assumptions of the Prandtl boundary layer theory, according to which the Navier-Stokes equations are reduced to a new system of "boundary layer" equations (see [2]). If we denote by (v_1 , v_2) the components of the velocity along the normals and parallels to a boundary line S in two-dimensional steady flow, and by $\partial/\partial\alpha$, $\partial/\partial\beta$ differentiations along these directions, then symbolically

$$\frac{\partial}{\partial\alpha} = O(R^{1/2}); \qquad \frac{\partial}{\partial\beta} = O(1), \tag{1.9}$$

$$v_1 = O(R^{-1/2}); \qquad v_2 = O(1), \tag{1.10}$$

and other physical quantities are $O(1)$. Thus we stretch the normal distance by a factor $R^{1/2}$, magnify v_1, the normal component of velocity, by $R^{1/2}$, and then consider that all new quantities and new derivatives have the same order of magnitude $O(1)$. Neglecting all but the terms of highest order in our equations, we obtain the reduced set of "boundary layer" equations. A solution of the latter equations constitutes an asymptotic integration of the Navier-Stokes equations. Now consider shock transition at S. Here, in contrast to (1.9), (1.10), we assume

$$\frac{\partial}{\partial\alpha} = O(R); \qquad \frac{\partial}{\partial\beta} = O(1), \tag{1.9'}$$

$$v_1 = O(1); \qquad v_2 = O(1), \tag{1.10'}$$

where these are suggested in the following manner. If we superpose a constant velocity $v_2 = O(1)$ on the motion discussed by von Mises, then we obtain a two-dimensional steady motion with S a straight line. For this motion, (1.9') and (1.10') clearly hold, in view of (1.5). Thus, in the general case, we stretch the normal distance by a factor R and then consider all new quantities and their new derivatives to have the same order of magnitude $O(1)$. Neglecting all but the terms of highest order in our equations, we obtain the reduced set of "shock transition" equations, a solution of which constitutes an asymptotic integration of the Navier-Stokes equations. These "shock transition" equations are found to be completely analogous to the system (1.1) through (1.4) and (1.5'). Hence the existence of "shock transition" regions in a real fluid can be inferred; further, the fact that as $R \to \infty$ these regions go over into discontinuity surfaces for which the Rankine-Hugoniot conditions of inviscid flow apply, can be shown.

Having discussed the points on which the boundary layer and the shock transition are analogous, we must next note two important differences. First, compressibility is an essential for the latter, but not for the former: shocks do not occur in an incompressible fluid. Second, time-variation of the motion produces essential changes in shock-transition, but not in the boundary layer. In the former, S has its own motion; in the latter, S is fixed. Hence the purpose of the paper is formulated precisely as follows: Given an arbitrary hypersurface S (which satisfies loose regularity conditions), a V^3 in (t, x, y, z) space it is possible to solve asymptotically the Navier-Stokes equations so that physical variables undergo rapid changes normal to S, and relatively slow changes parallel to S, in the neighborhood of S, provided that R is large enough. Moreover, in the limit $R \to \infty$, S becomes a discontinuity surface across which the Rankine-Hugoniot conditions are fulfilled. To make the discussion clearer, we shall work through two simpler cases first, one of steady motion and the other of unsteady.

Finally, we shall use the fact (inherent in von Mises' discussion) that du/ds, d^2u/ds^2, dT/ds, $d^2T/ds^2 \cdots$ are of the same order as u, $T \cdots$ in some fixed interval $[s]$, depending only on m, C_1, C_2, of s, and that the end points of this interval correspond very nearly to the Rankine-Hugoniot conditions which strictly hold for $s = \mp\infty$. This is equivalent to the result that the "thickness" of the shock is extremely small in von Mises' case, and that outside conditions are very nearly uniform.

2. Steady two-dimensional flow. The momentum equation in vector form may be written non-dimensionally:

$$\rho\left(\omega \times \mathbf{V} + \operatorname{grad} \frac{1}{2} V^2\right) = -\operatorname{grad} p + \frac{4}{3R} \operatorname{grad} \vartheta - \frac{1}{R} \operatorname{curl} \omega \tag{2.1}$$

where the vorticity ω and the dilatation ϑ are given by

$$\omega = \operatorname{curl} \mathbf{V}, \qquad \vartheta = \operatorname{div} \mathbf{V} \tag{2.2}$$

([1]—true also for compressible and heat-conducting fluid). In (2.1) R is the Reynold's number and the other quantities have their usual meaning. The equation of continuity may likewise be written:

$$\operatorname{div} \rho\mathbf{V} = 0 \tag{2.3}$$

and the equation of state:

$$p = \rho T. \tag{2.4}$$

The energy equation is not commonly derived under such general conditions but a short computation yields (appendix A):

$$\rho\mathbf{V} \cdot \operatorname{grad}\left(\frac{1}{2}V^2 + \frac{T}{\kappa - 1}\right) = \frac{\kappa - 1}{\kappa}\frac{1}{PR}\nabla^2 T - \operatorname{div}\left[\left(p + \frac{2}{3R}\vartheta\right)\mathbf{V}\right] + \frac{1}{R}\nabla^2 V^2 - \frac{1}{R}\operatorname{div}(\mathbf{V} \times \omega), \tag{2.5}$$

where P is the Prandtl number (assumed constant) and the dilatation ϑ is given in (2.2) together with the vorticity ω.

These are the equations of motion and we wish to refer them to a particular set of orthogonal curvilinear coordinates having coordinate lines normal and parallel to the

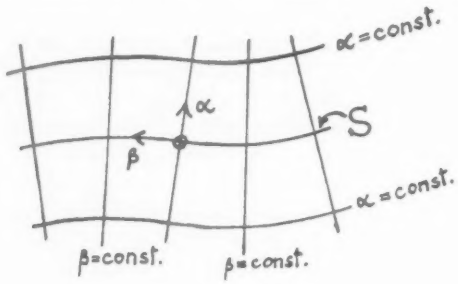

FIG. 1.

given line S, (Fig. 1). This system of coordinates was used by von Mises [2] to obtain the boundary layer equations for an arbitrary curved boundary line S. The square of the line element in these coordinates may be written

$$ds^2 = d\alpha^2 + \left(\frac{r + \alpha}{r}\right)^2 d\beta^2, \tag{2.6}$$

where β is the arc length on S measured from an arbitrary fixed point, $r = r(\beta)$ is the radius of curvature of S, and α is the normal distance from S. Hence if we write the components of velocity referred to these axes as (v_1, v_2) we have, instead of (2.1) and (2.2),

$$\rho\left\{v_1\frac{\partial v_1}{\partial\alpha}+\frac{r}{r+\alpha}v_2\frac{\partial v_1}{\partial\beta}-\frac{v_2^2}{r+\alpha}\right\}=-\frac{\partial p}{\partial\alpha}+\frac{4}{3R}\frac{\partial\vartheta}{\partial\alpha}-\frac{1}{R}\frac{r}{r+\alpha}\frac{\partial\omega}{\partial\beta}, \tag{2.7}$$

$$\rho\left\{v_1\frac{\partial v_2}{\partial\alpha}+\frac{r}{r+\alpha}v_2\frac{\partial v_2}{\partial\beta}+\frac{v_1v_2}{r+\alpha}\right\}=-\frac{r}{r+\alpha}\frac{\partial p}{\partial\beta}+\frac{4}{3R}\frac{r}{r+\alpha}\frac{\partial\vartheta}{\partial\beta}+\frac{1}{R}\frac{\partial\omega}{\partial\alpha}, \tag{2.8}$$

$$\vartheta=\frac{r}{r+\alpha}\left\{\frac{\partial}{\partial\alpha}\left(\frac{r+\alpha}{r}v_1\right)+\frac{\partial}{\partial\beta}(v_2)\right\},$$

$$\omega=\frac{r}{r+\alpha}\left\{\frac{\partial}{\partial\alpha}\left(\frac{r+\alpha}{r}v_2\right)-\frac{\partial}{\partial\beta}(v_1)\right\}, \tag{2.9}$$

and instead of the continuity equation (2.3),

$$\frac{\partial}{\partial\alpha}\left(\frac{r+\alpha}{r}\rho v_1\right)+\frac{\partial}{\partial\beta}(\rho v_2)=0 \tag{2.10}$$

The values of the various terms in the energy equation (2.5) may be obtained from

$$\mathbf{V}\cdot\operatorname{grad}\left(\frac{1}{2}V^2+\frac{T}{\kappa-1}\right)=\left(v_1\frac{\partial}{\partial\alpha}+v_2\frac{r}{r+\alpha}\frac{\partial}{\partial\beta}\right)\left(\frac{1}{2}V^2+\frac{T}{\kappa-1}\right) \tag{2.11a}$$

$$\nabla^2=\frac{r}{r+\alpha}\left\{\frac{\partial}{\partial\alpha}\left(\frac{r+\alpha}{r}\frac{\partial}{\partial\alpha}\right)+\frac{\partial}{\partial\beta}\left(\frac{r}{r+\alpha}\frac{\partial}{\partial\beta}\right)\right\} \tag{2.11b}$$

$$\mathbf{V}\times\boldsymbol{\omega}=(v_2\omega,\ -v_1\omega) \tag{2.11c}$$

with use of (2.9), i.e. we remember, for instance, that the expression for ϑ gives us the divergence of a vector $\mathbf{V}$.

In agreement with the ideas of Sec. 1, we now replace α by s/R and consider in our new system of coordinates that all functions and their derivatives are of the same order of magnitude. Then considering only the highest order terms in $1/R$, we may write the previous equations (2.7) through (2.10) as:

$$\rho v_1\frac{\partial v_1}{\partial s}=-\frac{\partial p}{\partial s}+\frac{4}{3}\frac{\partial}{\partial s}\left(\frac{\vartheta}{R}\right) \tag{2.7'}$$

$$\rho v_1\frac{\partial v_2}{\partial s}=\frac{\partial}{\partial s}\left(\frac{\omega}{R}\right) \tag{2.8'}$$

$$\frac{\vartheta}{R}=\frac{\partial v_1}{\partial s},\qquad\frac{\omega}{R}=\frac{\partial v_2}{\partial s} \tag{2.9'}$$

$$\frac{\partial}{\partial s}(\rho v_1)=0 \tag{2.10'}$$

and the energy equation, from (2.11a) through (2.11c), as

$$\rho v_1\frac{\partial}{\partial s}\left(\frac{1}{2}V^2+\frac{T}{\kappa-1}\right)=\frac{\kappa-1}{\kappa}\cdot\frac{1}{P}\frac{\partial^2T}{\partial s^2}-\frac{\partial}{\partial s}\left[\left(p+\frac{2\vartheta}{3R}\right)v_1\right]$$
$$+\frac{\partial^2}{\partial s^2}(V^2)-\frac{\partial}{\partial s}\left(v_2\frac{\omega}{R}\right) \tag{2.11'}$$

It is clear immediately that (2.8′) with (2.9′) is satisfied by

$$v_2 = v_2(\beta) \tag{2.8''}$$

which is the mathematical expression of the fact that the component of velocity parallel to the line S remains unaltered along a direction prependicular to S. The equations (2.7′), (2.10′) and (2.11′), with use of (2.9′) may be immediately integrated to give

$$\rho v_1 = m(\beta), \tag{2.10''}$$

$$mv_1 + p - \frac{4}{3}\frac{\partial v_1}{\partial s} = c_1(\beta), \tag{2.7''}$$

$$m\left[\frac{1}{2}V^2 + \frac{T}{\kappa - 1}\right] + v_1\left(p - \frac{4}{3}\frac{\partial v_1}{\partial s}\right) - \frac{\kappa - 1}{\kappa}\frac{1}{P}\frac{\partial T}{\partial s} = c_2(\beta), \tag{2.11''}$$

where we have made use of (2.8″). The equations (2.4), (2.7″), (2.10″) and (2.11″) are equivalent to (1.1) through (1.4) and (1.5′), with v_1 for u and c_1 , c_2 for mC_1 , mC_2 .

These equations have been obtained under the sole assumptions that (i) R is large,

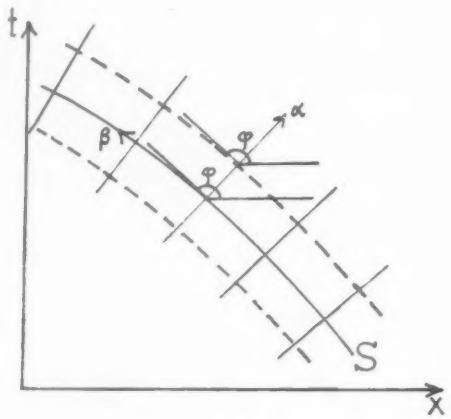

FIG. 2.

and (ii) changes along the shock are of the same order as $\partial u/\partial s$, $\partial^2 u/\partial s^2$, $\partial^2 T/\partial s^2$, etc. Thus they are accurate for the interval $[s]$ of section 1. Hence the existence of a shock is demonstrated. Letting $R \to \infty$, the equations become *exact* without assumption (ii) and $[s]$ may (in this case alone) be extended to

$$-\infty \leq s \leq +\infty \tag{2.12}$$

Thus S becomes a discontinuity and the Rankine-Hugoniot conditions hold.

3. Unsteady one-dimensional flow. The equations governing the motion are here, in non-dimensional form,

$$\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(\rho u^2 + p - \sigma) = 0, \tag{3.1}$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} = 0, \tag{3.2}$$

$$\frac{\partial}{\partial t}\left\{\rho\left(\frac{1}{2}u^2 + \frac{T}{\kappa - 1}\right)\right\} + \frac{\partial}{\partial x}\left\{\rho u\left(\frac{1}{2}u^2 + \frac{T}{\kappa - 1}\right) + H + (\sigma - p)u\right\} = 0 \tag{3.3}$$

with

$$\sigma = \frac{4}{3R}\frac{\partial u}{\partial x}, \qquad H = \frac{\kappa - 1}{\kappa}\frac{1}{PR}\frac{\partial T}{\partial x} \tag{3.3a}$$

and the equation of state is

$$p = \rho T. \tag{3.4}$$

Our given surface S reduces to a line in the (x, t) plane and we set up the same system of coordinates as in the previous section (Fig. 2). Thus α = const. represents a motion fixed relative to the "shock" S, and β = const. represents motion directly across S. Now equations (3.1) through (3.3a) may be written as

$$\frac{\partial A_1^i}{\partial \alpha}\sin\varphi - \frac{\partial A_2^i}{\partial \alpha}\cos\varphi + \frac{r}{r + \alpha}\left(\frac{\partial A_1^i}{\partial \beta}\cos\varphi + \frac{\partial A_2^i}{\partial \beta}\sin\varphi\right) = 0, \qquad i = 1, 2, 3 \tag{3.5}$$

$$\begin{aligned} A_1^1 &= \rho u^2 + p - \sigma, & A_2^1 &= \rho u, \\ A_1^2 &= \rho u, & A_2^2 &= \rho, \\ A_1^3 &= \rho u\left(\frac{1}{2}u^2 + \frac{T}{\kappa - 1}\right) + H + (\sigma - p)u, & A_2^3 &= \rho\left(\frac{1}{2}u^2 + \frac{T}{\kappa - 1}\right), \end{aligned} \tag{3.5a}$$

and

$$\begin{aligned} \sigma &= \frac{4}{3R}\left\{\frac{\partial u}{\partial \alpha}\sin\varphi + \frac{r}{r + \alpha}\frac{\partial u}{\partial \beta}\cos\varphi\right\}, \\ H &= \frac{(\kappa - 1)}{\kappa}\frac{1}{PR}\left\{\frac{\partial T}{\partial \alpha}\sin\varphi + \frac{r}{r + \alpha}\frac{\partial T}{\partial \beta}\cos\varphi\right\}, \end{aligned} \tag{3.6}$$

where we have used the rules for differentiation under change of variables. We note that

$$\varphi = \varphi(\beta) \tag{3.7}$$

We now replace α by s/R and consider that in the new system of coordinates all functions and their derivatives are of the same order of magnitude. Then considering only the highest order terms in $1/R$, we may write (3.5) and (3.6) in the form:

$$\frac{\partial}{\partial s}(A_1^i - A_2^i \cot\varphi) = 0, \qquad i = 1, 2, 3. \tag{3.5'}$$

and

$$\sigma = \frac{4}{3}\sin\varphi\frac{\partial u}{\partial s}, \qquad H = \frac{(\kappa - 1)}{\kappa}\frac{1}{PR}\sin\varphi\frac{\partial T}{\partial s}. \tag{3.6'}$$

If the velocity cot φ of the shock is written as c, the equations (3.5′) reduce to

$$mu + p - \sigma = c_1(\beta), \qquad \rho(u - c) = m(\beta),$$
$$m\left[\frac{1}{2}u^2 + \frac{T}{\kappa - 1}\right] - H + (p - \sigma)u = c_2(\beta). \tag{3.5″}$$

The equations (3.5″) and (3.6′) are equivalent to those of section 1 since (3.7) holds. Thus the discussion at the end of section 2 may be repeated.

4. Unsteady three-dimensional flow. In this, the most general case, the situation is a little more complicated. The differences between the present analysis and the preceding will be noted at each stage. First, the momentum equations in (t, x, y, z) space have no simple "vector" form which allows of simple transformation to new axes, as with (2.1) in (x, y) space. In fact, they may probably be dealt with best in the form:

$$\rho\left(\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \operatorname{grad} \mathbf{V}\right) = -\operatorname{grad} p + \frac{1}{3R} \operatorname{grad} \vartheta + \frac{1}{R} \nabla^2 \mathbf{V} \tag{4.1}$$

with

$$\vartheta = \operatorname{div} \mathbf{V}, \qquad \mathbf{V} = (u, v, w). \tag{4.1a}$$

The equation of continuity

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \rho \mathbf{V} = 0 \tag{4.2}$$

likewise has the same drawback. The equation of state

$$p = \rho T, \tag{4.3}$$

and the energy equation (Appendix A)

$$\frac{\partial I}{\partial t} + \operatorname{div} (I\mathbf{V}) = \frac{\kappa - 1}{\kappa} \frac{1}{PR} \nabla^2 T - \operatorname{div}\left[\left(p + \frac{2}{3R}\vartheta\right)\mathbf{V}\right] + \frac{1}{2R} \nabla^2 \mathbf{V}^2 + \frac{1}{R} \operatorname{div} (\mathbf{V} \cdot \operatorname{grad} \mathbf{V}) \tag{4.4}$$

with

$$\omega = \operatorname{curl} \mathbf{V}, \qquad I = \rho\left(\frac{1}{2} V^2 + \frac{T}{\kappa - 1}\right) \tag{4.5}$$

complete the system of equations governing the motion. The energy equation also is not of vector form.

Suppose that the Euclidean metric

$$ds^2 = dt^2 + dx^2 + dy^2 + dz^2 \tag{4.6}$$

is imposed on the (t, x, y, z) space. Then it is possible to set up a system of orthogonal curvilinear coordinates $(\alpha, \beta, \gamma, \delta)$ such that (4.6) reads:

$$ds^2 = d\alpha^2 + (h_1 + \lambda_1\alpha)^2\, d\beta^2 + (h_2 + \lambda_2\alpha)^2\, d\gamma^2 + (h_3 + \lambda_3\alpha)^2\, d\delta^2 \tag{4.7}$$

where the lines $\beta, \gamma, \delta = \text{const.}$ are normal to the given manifold S (here a V^3) and the

3-spaces α = const. are parallel to S (see Appendix A). Moreover h_1 , h_2 , h_3 , λ_1 , λ_2 , λ_3 , are functions of β, γ, δ alone, and

$$ds^2 = h_1^2\, d\beta^2 + h_2^2\, d\gamma^2 + h_3^2\, d\delta^2 \tag{4.8}$$

is the metric of S. To illustrate the nature of this system, we note that in the two-dimensional case of unsteady flow the lines (β, γ const.) are normal to a surface S whose lines of curvature are $\alpha = 0, \beta$ = const. and $\alpha = 0, \gamma$ = const. Also in this case h_1/λ_1 , h_2/λ_2 are the principal radii of curvature of S, and are functions of (β, γ).

We now wish to introduce a notation for vectors A_i in our 4-space. We shall denote by (A_1 , A_2 , A_3 , A_4)—round brackets—the components of A_i referred to the (t, x, y, z) axes, and by $\{A_1' , A_2' , A_3' , A_4'\}$—curled brackets—its components referred to the $(\alpha, \beta, \gamma, \delta)$ axes. Moreover we shall denote by div* the divergence i.e.

$$\text{div}^*\, A_i = \frac{\partial A_1}{\partial t} + \frac{\partial A_2}{\partial x} + \frac{\partial A_3}{\partial y} + \frac{\partial A_4}{\partial z} \tag{4.9}$$

and similarly the gradient by grad*. Finally we shall suppose that the relative orientation of the two sets of axes is given by the scheme

	t	x	y	z
α	k_1	l_2	m_3	n_4
β	k_1	l_2	m_3	n_4
γ	k_1	l_2	m_3	n_4
δ	k_1	l_2	m_3	n_4

(4.10)

in which all elements are independent of α.

Consider now the first of the momentum equations (4.1). We have

$$\begin{aligned} \frac{\partial u}{\partial t} + \mathbf{V} \cdot \text{grad}\, u &= (l, u, v, w) \cdot \text{grad}^*\, u \\ &= \{v_1 , v_2 , v_3 , v_4\} \cdot \left\{\frac{\partial u}{\partial \alpha}, \frac{1}{h_1 + \lambda_1 \alpha}\frac{\partial u}{\partial \beta}, \frac{1}{h_2 + \lambda_2 \alpha}\frac{\partial u}{\partial \gamma}, \frac{1}{h_3 + \lambda_3 \alpha}\frac{\partial u}{\partial \delta}\right\}, \end{aligned} \tag{4.11}$$

where

$$\begin{aligned} v_1 &= k_1 + l_1 u + m_1 v + n_1 w, \qquad v_2 = k_2 + l_2 u + m_2 v + n_2 w, \\ v_3 &= k_3 + l_3 u + m_3 v + n_3 w, \qquad v_4 = k_4 + l_4 u + m_4 v + n_4 w. \end{aligned} \tag{4.11a}$$

As in the previous sections we now replace α by s/R and (considering all functions and derivatives to be of the same order of magnitude in the new coordinates) retain only the lowest powers of $1/R$. Thus:

$$\frac{\partial u}{\partial t} + \mathbf{V} \cdot \text{grad}\, u = R v_1 \frac{\partial u}{\partial s} \tag{4.12}$$

Also, to the same order,

$$\frac{\partial p}{\partial x} = Rl_1 \frac{\partial p}{\partial s}, \qquad \frac{\partial \vartheta}{\partial x} = Rl_1 \frac{\partial \vartheta}{\partial s}. \tag{4.13}$$

Further we have

$$\operatorname{div}^* \left(0, \frac{\partial u}{\partial x}, 0, 0\right) = \operatorname{div}^* \{l_1 U, l_2 U, l_3 U, l_4 U\}$$

with

$$U = l_1 \frac{\partial u}{\partial \alpha} + \frac{l_2}{h_1 + \lambda_1 \alpha} \frac{\partial u}{\partial \beta} + \frac{l_3}{h_2 + \lambda_2 \alpha} \frac{\partial u}{\partial \gamma} + \frac{l_4}{h_3 + \lambda_3 \alpha} \frac{\partial u}{\partial \delta},$$

so that

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{(h_1 + \lambda_1\alpha)(h_2 + \lambda_2\alpha)(h_3 + \lambda_3\alpha)} \left\{ \frac{\partial}{\partial \alpha} [(h_1 + \lambda_1\alpha)(h_2 + \lambda_2\alpha)(h_3 + \lambda_3\alpha) l_1 U] \right.$$

$$\left. + \frac{\partial}{\partial \beta} [(h_2 + \lambda_2\alpha)(h_3 + \lambda_3\alpha) l_2 U] + \cdots + \cdots \right\} \tag{4.14}$$

$$= R^2 l_1^2 \frac{\partial^2 u}{\partial s^2},$$

to the highest order. We thus have from (4.14)

$$\nabla^2 u = R^2 (1 - k_1^2) \frac{\partial^2 u}{\partial s^2} \tag{4.15}$$

To complete the discussion of (4.1) we have from (4.1a):

$$\vartheta = \operatorname{div}^* (0, u, v, w) = \operatorname{div}^* \{v_1 - k_1, v_2 - k_2, v_3 - k_3, v_4 - k_4\}$$

$$= R \frac{\partial}{\partial s} (v_1 - k_1) \tag{4.16}$$

The x momentum equation, in $(\alpha, \beta, \gamma, \delta)$ coordinates, is thus,

$$\rho v_1 \frac{\partial u}{\partial s} = -l_1 \frac{\partial p}{\partial s} + \frac{1}{3} l_1 \frac{\partial^2}{\partial s^2} (v_1 - k_1) + (1 - k_1^2) \frac{\partial^2 u}{\partial s^2}. \tag{4.17a}$$

The y and z momentum equations are obtained in a similar manner: we merely change u to v and w, and l_1 to m_1 and n_1, respectively, to form two new equations (4.17b) and (4.17c). Unlike the case before, these equations are not component equations referred to the new system $(\alpha, \beta, \gamma, \delta)$ but are referred to the old system in new system terms.

The continuity equation (4.2) can be written as

$$0 = \operatorname{div}^* (\rho, \rho u, \rho v, \rho w) = \operatorname{div}^* \{\rho v_1, \rho v_2, \rho v_3, \rho v_4\}$$

and this becomes

$$\frac{\partial}{\partial s} (\rho v_1) = 0. \tag{4.18}$$

From the preceding discussion we may list immediately the following reductions of the terms of the energy equation (4.4)

$$\frac{\partial I}{\partial t} + \operatorname{div}(I\mathbf{V}) = R\frac{\partial}{\partial s}(Iv_1), \tag{4.19a}$$

$$\nabla^2 T = R^2(1 - k_1^2)\frac{\partial^2 T}{\partial s^2}, \tag{4.19b}$$

$$\nabla^2(V^2) = R^2(1 - k_1^2)\frac{\partial^2(V^2)}{\partial s^2}, \tag{4.19c}$$

$$\operatorname{div}\left(p + \frac{2\vartheta}{3R}\right)\mathbf{V} = R\frac{\partial}{\partial s}\left[\left(p + \frac{2\vartheta}{3R}\right)(v_1 - k_1)\right]. \tag{4.19d}$$

Moreover similarly to (4.12)

$$\mathbf{V} \cdot \operatorname{grad}\mathbf{V} = R(v_1 - k_1)\frac{\partial \mathbf{V}}{\partial s}$$

so that similarly to (4.16)

$$\begin{aligned}\operatorname{div}(\mathbf{V} \cdot \operatorname{grad}\mathbf{V}) &= R^2\frac{\partial}{\partial s}\left\{(v_1 - k_1)\frac{\partial}{\partial s}(v_1 - k_1)\right\} \\ &= R^2\frac{\partial^2}{\partial s^2}\left\{\frac{1}{2}(v_1 - k_1)^2\right\}.\end{aligned} \tag{4.19e}$$

Using (4.19a) through (4.19e) and (4.16), we may replace (4.4) by

$$\begin{aligned}\frac{\partial}{\partial s}\left[\rho v_1\left(\frac{1}{2}V^2 + \frac{T}{\kappa - 1}\right)\right] &= \frac{\kappa - 1}{\kappa}\frac{1}{P}(1 - k_1^2)\frac{\partial^2 T}{\partial s^2} - \frac{\partial}{\partial s}[p(v_1 - k_1)] \\ &\quad + \frac{1}{2}(1 - k_1^2)\frac{\partial^2(V^2)}{\partial s^2} + \frac{1}{6}\frac{\partial^2}{\partial s^2}(v_1 - k_1)^2.\end{aligned} \tag{4.20}$$

Before discussing these equations, we must introduce simplifications. If we write

$$D^2 = l_1^2 + m_1^2 + n_1^2 = 1 - k_1^2$$

it is clear that $\mathbf{n} = (l_1/D, m_1/D, n_1/D)$ is the unit normal in (x, y, z) space to the instantaneous position of the "shock." Moreover $-k_1/D$ is the normal velocity of the "shock" in this space. Denoting then the normal component of velocity by u_n and the normal velocity of the "shock" by U i.e.

$$l_1 u + m_1 v + n_1 w = Du_n, \qquad -k_1 = DU. \tag{4.21}$$

We may rewrite (4.17a), (4.17b), (4.17c), (4.18) and (4.20) in the more familiar form:

$$\rho(u_n - U)\frac{\partial u}{\partial s} = \frac{-l_1}{D}\frac{\partial p}{\partial s} + \frac{1}{3}l_1\frac{\partial^2 u_n}{\partial s^2} + D\frac{\partial^2 u}{\partial s^2}, \tag{4.17'a}$$

$$\rho(u_n - U)\frac{\partial v}{\partial s} = \frac{-m_1}{D}\frac{\partial p}{\partial s} + \frac{1}{3}m_1\frac{\partial^2 u_n}{\partial s^2} + D\frac{\partial^2 v}{\partial s^2}, \tag{4.17'b}$$

$$\rho(u_n - U)\frac{\partial w}{\partial s} = \frac{-n_1}{D}\frac{\partial p}{\partial s} + \frac{1}{3}n_1\frac{\partial u_n^2}{\partial s^2} + D\frac{\partial^2 w}{\partial s^2}, \tag{4.17'c}$$

$$\frac{\partial}{\partial s}[\rho(u_n - U)] = 0 \tag{4.18'}$$

$$\frac{\partial}{\partial s}\left[\rho(u_n - U)\left(\frac{1}{2}V^2 + \frac{T}{\kappa - 1}\right)\right] = \frac{\kappa - 1}{\kappa}\frac{D}{P}\frac{\partial^2 T}{\partial s^2} - \frac{\partial(pu_n)}{\partial s} + \frac{D}{2}\frac{\partial^2(V^2)}{\partial s^2} + \frac{D}{6}\frac{\partial^2}{\partial s^2}(u_n^2) \tag{4.20'}$$

If we multiply (4.17'a) by l_1, (4.17'b) by m_1, and (4.17'c) by n_1 and add we obtain

$$\rho(u_n - U)\frac{\partial u_n}{\partial s} = -\frac{\partial p}{\partial s} + \frac{4D}{3}\frac{\partial^2(u_n)}{\partial s^2} \tag{4.21}$$

which may be integrated by virtue of (4.18'), i.e.

$$\rho(u_n - U) = m(\beta, \gamma, \delta), \tag{4.22}$$

to give

$$mu_n + p - \sigma = c_1(\beta, \gamma, \delta) \tag{4.23}$$

with

$$\sigma = \frac{4}{3}D\frac{\partial(u_n)}{\partial s} \tag{4.23a}$$

Furthermore if we multiply (4.17'a) by m_1, (4.17'b) by l_1 and subtract we have

$$m\frac{\partial}{\partial s}(m_1 u - l_1 v) = D\frac{\partial^2}{\partial s^2}(m_1 u - l_1 v)$$

which is satisfied if

$$\frac{1}{D}(m_1 u - l_1 v) = f(\beta, \gamma, \delta). \tag{4.24}$$

Similarly the other components of $\mathbf{V} \times \mathbf{n}$ are independent of α. Hence the component of $\mathbf{V}$ tangential to the instantaneous position of the shock in the (x, y, z) is independent of α. Thus we may write

$$\frac{\partial}{\partial s}(V^2) = \frac{\partial}{\partial s}(u_n^2) \tag{4.25}$$

and the energy equation (4.20') can be written in integrated form:

$$m\left(\frac{1}{2}u_n^2 + \frac{T}{\kappa - 1}\right) - H + (p - \sigma)u_n = c_2(\beta, \gamma, \delta) \tag{4.26}$$

with

$$H = \frac{\kappa - 1}{\kappa} \frac{D}{P} \frac{\partial T}{\partial s} \tag{4.26a}$$

The equations (4.3), (4.22), (4.23), (4.23a), (4.26) and (4.26a) are equivalent to those of section 1 with u_n for u, and definition of $m(\beta, \gamma, \delta)$ slightly changed, since $D = D(\beta, \gamma, \delta)$ from (4.10). Hence discussion at end of section 2 applies equally well here.

Summary. Given an arbitrary hypersurface S in (t, x, y, z) space and an arbitrary distribution (within the loose restriction of the constant c of von Mises' paper) of $\mathbf{V}$, T, ρ and derivatives on it, we have the following:

(*i*) It is possible to construct a family of solutions of the Navier-Stokes equations having large normal derivatives on S and (comparatively) small derivatives parallel to S, provided μ is small enough. The solution is analogous to a solution of the steady one-dimensional equations of motion, in which the x-differentiation is replaced by

$$\frac{\partial}{\partial n} - U \frac{\partial}{\partial t},$$

where U is the normal velocity of the "instantaneous" section of S by (x, y, z) space and $\partial/\partial n$ denotes differentiation along the normal to this section. Moreover the Rankine-Hugoniot conditions are approximated very nearly, and the component of $\mathbf{V}$ in the tangent plane to the "instantaneous" section of S by (x, y, z) space is conserved along the normal direction of S.

(*ii*) As $\mu \to 0$ the surface S becomes a discontinuity surface and the Rankine-Hugoniot conditions hold exactly.

The purpose of this paper, as stated at the beginning, was to prove the existence of shocks, using as sole basis the Navier-Stokes equations for the motion of a perfect fluid. As a by-product of the analysis, we are given a method of investigating the structure of a shock in the most general motion of a perfect fluid. However, if such an application is made, it must be borne in mind that it will only be valid when the Navier-Stokes equations hold, that is when the fluid can be considered as a continuum, in other words, for weak shocks.

Appendix A

If we denote an element of the stress-tensor of a viscous fluid by σ_{ij} and an element of the rate of strain tensor by e_{ij}, the generalized concept of viscosity leads to [3]

$$\sigma_{ij} = -(p + 2/3\ \mu\vartheta)\delta_{ij} + \mu e_{ij} \tag{A.1}$$

with

$$e_{ij} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}, \qquad \vartheta = \frac{1}{2} e_{ii} = \operatorname{div} \mathbf{V}. \tag{A.2}$$

Here we use dimensional quantities, with μ the coefficient of viscosity and $(x, y, z) = (x_1, x_2, x_3)$, $(u, v, w) = (u_1, u_2, u_3)$. Since the total energy [4] is

$$I = \rho\left(\frac{1}{2} V^2 + \frac{\overline{R}T}{\kappa - 1}\right), \tag{A.3}$$

the energy equation is easily seen to be

$$\rho \frac{D}{Dt}\left(\frac{1}{2} V^2 + \frac{\bar{R}T}{\kappa - 1}\right) = k\nabla^2 T + \frac{\partial(u_i \sigma_{ij})}{\partial x_j} \tag{A.4}$$

From (A.1) and (A.2) it follows, however, that

$$u_i \sigma_{ij} = -\left(p + \frac{2}{3}\mu\vartheta\right)u_i + \frac{\partial}{\partial x_j}\left(\frac{1}{2} V^2\right) + u_i \frac{\partial u_j}{\partial x_i} \cdot \tag{A.5}$$

Inserting this into the right hand side of (A.4) and using the equation of continuity (4.2) on the left hand side, we obtain the dimensional form of (4.4).

For one-dimensional unsteady flow, we have $\mathbf{V} = (u, 0, 0)$ and

$$-\frac{2}{3}\mu \operatorname{div}(\vartheta \mathbf{V}) + \frac{\mu}{2}\nabla^2 \mathbf{V}^2 + \mu \operatorname{div}(\mathbf{V} \cdot \operatorname{grad} \mathbf{V}) = \frac{4}{3}\mu \frac{\partial}{\partial x}\left(u \frac{\partial u}{\partial x}\right), \tag{A.6}$$

and (4.2) reduces to (3.3).

For steady two dimensional flow $\partial/\partial t = 0$ and the transformations of vector analysis yield

$$-\frac{1}{2}\mu\nabla^2 V^2 + \mu \operatorname{div}(\mathbf{V} \cdot \operatorname{grad} \mathbf{V}) = \mu \operatorname{div}(\boldsymbol{\omega} \times \mathbf{V}). \tag{A.7}$$

Hence (A.7) reduces (4.2) to the form (2.5).

Appendix B

Let $\mathbf{r} = \mathbf{r}(\nu_1, \nu_2, \nu_3)$ be the position vector of the manifold S (V^3) in the (t, x, y, z)-space, where (ν_1, ν_2, ν_3) form an orthogonal system of coordinates, i.e.

$$\mathbf{r}_i \cdot \mathbf{r}_j = |\mathbf{r}_i|\,|\mathbf{r}_j|\,\delta_{ij} \tag{B.1}$$

with

$$\mathbf{r}_i = \frac{\partial \mathbf{r}}{\partial \nu_i} \qquad (i = 1, 2, 3)$$

The unit normal $\mathbf{n}$ to S is uniquely determined by the equations:

$$\mathbf{n} \cdot \mathbf{r}_i = 0, \tag{B.2}$$

$$\mathbf{n} \cdot \mathbf{n} = 1. \tag{B.3}$$

From (B.3) we have by differentiation:

$$\mathbf{n} \cdot \mathbf{n}_i = 0, \tag{B.4}$$

and hence we may write

$$\mathbf{n}_i = \sum_{j=1}^{3} n_{ij} \frac{|\mathbf{r}_i|}{|\mathbf{r}_j|} \mathbf{r}_j \tag{B.5}$$

with suitable $n_{ij} = n_{ij}(\nu_1, \nu_2, \nu_3)$. Moreover from (B.2) we have

$$\mathbf{n}_i \cdot \mathbf{r}_i + \mathbf{n} \cdot \mathbf{r}_{ij} = 0,$$

whence

$$\mathbf{n}_i \cdot \mathbf{r}_i = \mathbf{n}_i \cdot \mathbf{r}_j \tag{B.6}$$

or substituting (B.5) and noting (B.1)

$$n_{ij} = n_{ji}\,. \tag{B.7}$$

Now suppose we have a unit vector in S defined by

$$\mathbf{s} = \sum_{i=1}^{3} s_i \frac{\mathbf{r}_i}{|\mathbf{r}_i|} \tag{B.8}$$

Then the derivative of $\mathbf{n}$ in the direction $\mathbf{s}$ is

$$\sum_{i=1}^{3} \mathbf{n}_i \frac{s_i}{|\mathbf{r}_i|} = \sum_{i=1}^{3} \sum_{j=1}^{3} n_{ij} s_i \frac{\mathbf{r}_j}{|\mathbf{r}_i|} \tag{B.9}$$

and this lies along $\mathbf{s}$ if

$$\sum_{i=1}^{3} n_{ij} s_i = k s_j \qquad (j = 1, 2, 3) \tag{B.10}$$

for some k. From the theory of matrices it is known [5] that since n_{ij} is symmetric (B.7) there exists at least one set of mutually orthogonal real unit eigenvectors $\mathbf{s}^{(\lambda)} (\lambda = 1, 2, 3)$ satisfying (B.10) with corresponding real eigenvalues $k = k^{(\lambda)}$. Taking these directions $\mathbf{s}^{(\lambda)}$ as the coordinate directions of a new set of orthogonal coordinates (β, γ, δ) in S we have

$$\mathbf{n}_\beta = \mu_1 \mathbf{r}_\beta\,, \qquad \mathbf{n}_\gamma = \mu_2 \mathbf{r}_\gamma\,, \qquad \mathbf{n}_\delta = \mu_3 \mathbf{r}_\delta \tag{B.11}$$

with μ_1, μ_2, μ_3 functions of (β, γ, δ).

Now let $\mathbf{n}$ complete a system of four orthogonal directions in 4-space and α be distance measured along it. Then if $\mathbf{R}$ is the position vector in this space

$$\mathbf{R} = \mathbf{r} + \alpha \mathbf{n}$$

with $\mathbf{r}$, $\mathbf{n}$ independent of α. Hence, using (B.11), we have

$$\begin{aligned} &\mathbf{R}_\alpha = \mathbf{n}, \qquad \mathbf{R}_\beta = \mathbf{r}_\beta + \alpha \mathbf{n}_\beta = \mathbf{r}_\beta (1 + \mu_1 \alpha), \\ &\mathbf{R}_\gamma = \mathbf{r}_\gamma (1 + \mu_2 \alpha), \qquad \mathbf{R}_\delta = \mathbf{r}_\delta (1 + \mu_3 \alpha). \end{aligned} \tag{B.12}$$

Hence $(\alpha, \beta, \gamma, \delta)$ form a system of orthogonal curvilinear coordinates with

$$ds^2 = d\alpha^2 + (h_1 + \lambda_1 \alpha)^2 \, d\beta^2 + (h_2 + \lambda_2 \alpha)^2 \, d\gamma^2 + (h_3 + \lambda_3 \alpha)^2 \, d\delta^2, \tag{B.13}$$

where $h_1^2 = \mathbf{r}_\beta^2$ etc., independent of α, and $\lambda_1 = h_1 \mu_1$ etc., independent of α.

We have thus demonstrated the validity of (4.7) and (4.8) as well as the statements about these equations. We have constructed V^3-spaces $\alpha = \text{const.}$ parallel to S (in the obvious sense of the term) and used them to form an orthogonal system of coordinates.

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ON DOUBLE-PULSE STABILITY CRITERIA WITH DAMPING*

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Summary. A simple necessary condition for the stability of a linear dynamic system with elastic and damping characteristics which vary periodically with the same period is derived. General explicit necessary and sufficient conditions for stability are then developed for a double-pulse system. Such a system can be characterized by a pair of eigenvalues, or complex frequencies, corresponding to each half-period, and the stability of this system depends only on these complex frequencies. It is shown that a necessary, though not sufficient, condition for the stability of any such system is that the arithmetic mean of the real parts of all four of the complex frequencies over an entire period be negative or zero. This is shown to be true, more generally, for an n-pulse system. The physical significance of the results is discussed, and numerical examples are given.

Introduction. A variety of physical phenomena[1] are characterized by a linear homogeneous differential equation with periodic coefficients. Sometimes the chief interest is in the stability of the solutions of such equations, rather than in the solutions themselves. If the system which these equations characterize is such that the coefficients vary (periodically) by changing discontinuously from one constant value to another one or

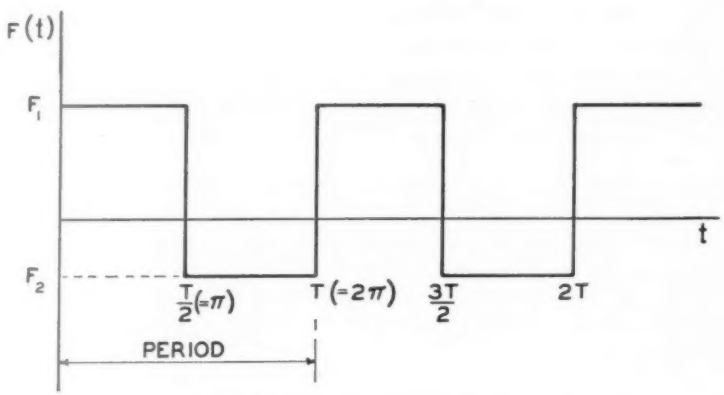

FIG. 1. Double-Pulse Function $F(t)$ with Period T $(= 2\pi)$.

more times during a period of variation (Figs. 1 and 2) then an exact solution of the equations can in principle be obtained without difficulty. Coefficients varying in the manner of Figs. 1 and 2 are called "rectangular pulses". In particular, if the coefficient

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[1]Examples are given in: (a) B. Van der Pol & M. J. O. Strutt, *On the stability of the solutions of Mathieu's equation*, Phil. Mag., **5**, 18 (1928); (b) S. Timoshenko, *Vibration problems in engineering*, second edition, D. Van Nostrand Co., N. Y., 1937, Chapter III; (c) J. P. Den Hartog, *Mechanical vibrations*, third edition, McGraw-Hill Book Co., N. Y., 1947, p. 408.

assumes only two different constant values, one for each half-period of variation (Fig. 1) it is called a "double-pulse". In cases in which the periodic coefficients do not vary as rectangular pulses, it is often possible to obtain approximate or at least qualitative solutions by replacing these coefficients by either double pulses or four pulses.[2] This type of approximation has, in fact, been recently applied.[3]

The purpose of this paper is to present explicit stability criteria for systems which can be represented by equations with periodic, especially double-pulse, coefficients for

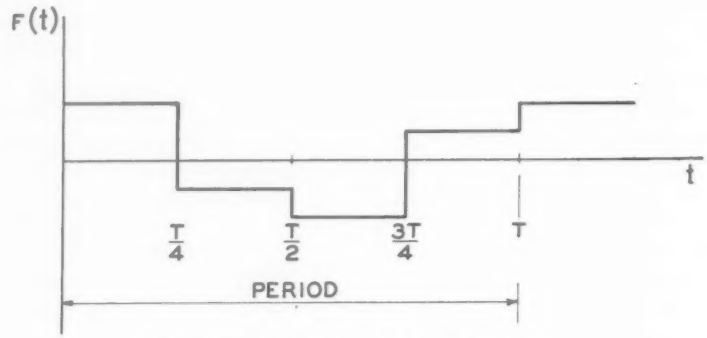

FIG. 2. Four-Pulse Function $F(t)$ with Period T.

the damping as well as for the elastic characteristics. Such a system without damping has been analyzed in detail by Meissner[4] and by Van der Pol and Strutt.[5] Schwerin[6] subsequently extended this investigation to systems with linearly varying characteristics. A particular double-pulse system with damping has also been investigated,[7] without, however, attempts to derive explicit equations and conclusions for damped systems in general.

The results obtained in the present analysis serve to present useful theorems of physical interest, to demonstrate some significant conclusions which might not be readily foreseen without the mathematical analysis, and to facilitate any actual stability calculations based on the double-pulse method.

General theory. The free motion of a linear dynamic system with spring stiffness k and damping coefficient f varying periodically with time, with period 2π, can be represented by the equation:

$$x'' + f(t)x' + k(t)x = 0 \tag{1}$$

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[2]B. Van der Pol and M. J. O. Strutt, op. cit.

[3]G. Horvay and S. W. Yuan, *Stability of rotor blade flapping motion when the hinges are tilted. Generalization of the "rectangular ripple" method of solution*, J. Aero. Sci., 14, 583-593, (1947). Also, S. W. Yuan and M. Morduchow, *On the stability of the transient motion of helicopter blades in flapping and lagging*, Reissner Anniv. Vol., J. W. Edwards, 1949, p. 163.

[4]E. Meissner, *Über Schättelschwingungen in Systemen periodisch veränderlicher Elastizität*, Schweizer. Bauzeitung, 1918, p. 95 (1918).

[5]op. cit.

[6]E. Schwerin, *Ein allgemeines Integrationsverfahren für quasiharmonische Schwingungsvorgänge*, Z. Techn. Phys. 12, 104 (1931).

[7]G. Horvay and S. W. Yuan, op. cit. Also, S. W. Yuan and M. Morduchow, op. cit.

where $x(t)$ denotes displacement, the primes denote differentiation with respect to the time t, and $f(t)$ and $k(t)$ are functions having the period 2π. By letting

$$x = v(t) \exp\left(-\frac{1}{2}\int f\,dt\right), \tag{2}$$

Eq. (1) can be transformed into the form

$$v'' + k_1(t)v = 0, \tag{3}$$

where $k_1(t)$ has the period 2π. By Floquet's theory[8] at least one particular integral of Eq. (3) is of the form

$$v_1 = \phi(t) \exp(\mu t) \tag{4a}$$

where $\phi(t)$ is a function with the period 2π, and μ is a definite constant. From this it can be shown that a second independent integral of Eq. (3) must have the form

$$v_2 = \psi(t) \exp(-\mu t) \tag{4b}$$

where $\psi(t)$ has the period 2π.

From Eq. (2), and from the fact that, due to the periodicity of $f(t)$,

$$\int_t^{t+2\pi} f(t)\,dt = \text{constant} = C, \tag{5}$$

say, it follows that the solutions $x_1(t)$ and $x_2(t)$ of Eq. (1), corresponding to v_1 and v_2 respectively, will satisfy the following relations, regardless of the time t:

$$x_1(t + 2\pi)/x_1(t) = \sigma_1\,, \qquad x_2(t + 2\pi)/x_2(t) = \sigma_2 \tag{6a}$$

where σ_1 and σ_2 are constants given by

$$\sigma_1 = \exp(2\pi\mu - \tfrac{1}{2}C), \qquad \sigma_2 = \exp(-2\pi\mu - \tfrac{1}{2}C). \tag{6b}$$

The general solution of Eq. (1) can be written as

$$x = A_1x_1(t) + A_2x_2(t), \tag{7}$$

where A_1 and A_2 are arbitrary constants.

The dynamic system may be defined as stable when both of its component free motions x_1 and x_2 subside with time, and unstable if either of these motions amplifies with time. From Eqs. (6a) it then follows that the system will be stable if $|\sigma_1| < 1$ and $|\sigma_2| < 1$, while the system will be unstable if either $|\sigma_1| > 1$ or $|\sigma_2| > 1$. The case $|\sigma_1| = 1$ and $|\sigma_2| = 1$ may be defined as one of neutral stability.

Equations (6b) imply that $\sigma_1\sigma_2 = \exp(-C)$. Hence in order that both $|\sigma_1| \leq 1$ and $|\sigma_2| \leq 1$, it is necessary that $C \geq 0$. This can be expressed by the following theorem:

THEOREM I: *A necessary condition for the stability of a dynamic system characterized by an equation of the form* (1) *is that*

$$\int_0^{2\pi} f(t)\,dt \geq 0$$

[8] M. G. Floquet, *Sur les équations différentielles linéaires à coefficients périodiques*, Ann. Ec. Norm. 12, 47 (1883). Also, E. T. Whittaker and G. N. Watson, *Modern analysis*, Macmillan Co., N. Y., 1948, pp. 412-413.

Physically this condition can be interpreted as requiring that the average value of the periodic damping coefficient $f(t)$ during any period must be non-negative. Thus, for stability, there must in the average be a positive damping force.

Double-pulse system. To investigate, in particular, the stability of a double-pulse system with damping, let $f(t)$ and $k(t)$ vary periodically in the manner of Fig. 1, i.e. let $f = f_1$ and $k = k_1$ for $0 < t < \pi$, while $f = f_2$ and $k = k_2$ for $\pi < t < 2\pi$; the constants f_1 , f_2 , k_1 and k_2 may be positive or negative. Then Eq. (1) can be solved for each of the two time-intervals as a linear equation with constant coefficients. The solution can be expressed as:

$$\begin{aligned} x &= C_1 \exp(p_1 t) + \overline{C}_1 \exp(\overline{p}_1 t), \qquad 0 \leq t \leq \pi \\ x &= C_2 \exp(p_2 t) + \overline{C}_2 \exp(\overline{p}_2 t), \qquad \pi \leq t \leq 2\pi \end{aligned} \tag{8}$$

where the C_i and $\overline{C}_i$ are arbitrary constants, the p_i and $\overline{p}_i$ are constants given by:

$$p_i = -\frac{f_i}{2} + \left[\left(\frac{f_i}{2}\right)^2 - k_i\right]^{1/2}, \qquad \overline{p}_i = -\frac{f_i}{2} - \left[\left(\frac{f_i}{2}\right)^2 - k_i\right]^{1/2} \tag{9}$$

The p_i and $\overline{p}_i$ are essentially eigenvalues corresponding to each half-period of variation of the damping and elastic coefficients, and are sometimes called "complex frequencies". Their physical interpretation follows from Eqs. (8). Thus, if a system were characterized by constant dynamical coefficients leading to a (single) pair of complex conjugate values for p and $\overline{p}$, then the system would oscillate freely with a natural frequency proportional to the imaginary part of p or $\overline{p}$, and would be damped out with a logarithmic decrement directly proportional to the negative of the real part and inversely proportional to the imaginary part of p or $\overline{p}$. A system with constant coefficients characterized by a single pair of real values of p and $\overline{p}$ would be non-oscillatory and would be stable if both p and $\overline{p}$ are negative, but unstable if either p or $\overline{p}$ is positive. For convenience, the negatives of the real parts of p_i and $\overline{p}_i$ will here be called "damping factors".

The arbitrary constants C_i and $\overline{C}_i$ in Eqs. (8) are related by the two conditions of continuity of displacement x and velocity $\dot{x}$ at $t = \pi$. Moreover, the following two additional conditions can be prescribed:

$$(x)_{t=2\pi} = \sigma(x)_{t=0}\,, \qquad (x')_{t=2\pi} = \sigma(x')_{t=0}\,, \tag{10}$$

where σ is a constant, to be determined. It can be shown from Eqs. (2), (4a, b) and (7) that by satisfying conditions (10), initial conditions are implicitly prescribed so that A_2 or A_1 in Eq. (7) will be zero,[9] and so that consequently the two values of σ which will thus be obtained will correspond exactly to σ_1 and σ_2 respectively of Eqs. (6b). By applying the continuity conditions and conditions (10) to Eqs. (8) a set of four linear homogeneous equations in C_1 , $\overline{C}_1$, C_2 and $\overline{C}_2$ is obtained. The condition for the existence of a non-trivial solution to these equations leads to a determinantal equation, which when expanded can be written in the form:

$$\sigma^2 + N\sigma + M = 0, \tag{11}$$

where

[9] This fact, although implied, often appears not to be clearly stated in the literature.

$$N = \frac{(p_1 - \bar{p}_2)(p_2 - p_1)(P_1\bar{P}_2 + \bar{P}_1P_2) + (p_2 - \bar{p}_1)(\bar{p}_2 - p_1)(P_1P_2 + \bar{P}_1\bar{P}_2)}{(\bar{p}_1 - p_1)(\bar{p}_2 - p_2)}, \tag{12a}$$

$$M = P_1\bar{P}_1P_2\bar{P}_2, \tag{12b}$$

$$P_i = \exp(p_i\pi), \qquad \bar{P}_i = \exp(\bar{p}_i\pi).$$

The roots

$$\sigma = -\frac{N}{2} \pm \left[\left(\frac{N}{2}\right)^2 - M\right]^{1/2} \tag{13}$$

of Eq. (11) then determine the stability of the system.

Three general cases of double-pulse systems, depending on whether the eigenvalues p_i and $\bar{p}_i$ are real or complex, can be distinguished. By substitution into Eqs. (12a, b) the following expressions for N and M in these various cases are obtained (a_i , b_i , c_i and d_i denote real positive or negative constants, while $i = (-1)^{1/2}$).

CASE I. *Two pairs of complex conjugate eigenvalues*:

$$p_1 = -a_1 + b_1i, \qquad \bar{p}_1 = -a_1 - b_1i, \qquad p_2 = -a_2 + b_2i, \qquad \bar{p}_2 = -a_2 - b_2i.$$

$$\begin{aligned} 2b_1b_2N = (\exp[-\pi(a_1 + a_2)])\{&[(a_1 - a_2)^2 + (b_1 - b_2)^2]\cos(b_1 - b_2)\pi \\ &- [(a_1 - a_2)^2 + (b_1 + b_2)^2]\cos(b_1 + b_2)\pi\} \end{aligned} \tag{14a}$$

$$M = \exp[-2\pi(a_1 + a_2)] \tag{14b}$$

CASE II. *Two pairs of real eigenvalues*:

$$p_1 = -c_1, \qquad \bar{p}_1 = -d_1, \qquad p_2 = -c_2, \qquad \bar{p}_2 = -d_2.$$

$$\begin{aligned} (c_1 - d_1)(c_2 - d_2)N = (d_1 - d_2)(c_2 - c_1)\{&\exp[-\pi(c_1 + d_2)] \\ &+ \exp[-\pi(c_2 + d_1)]\} \end{aligned} \tag{15a}$$

$$+ (c_2 - d_1)(d_2 - c_1)\{\exp[-\pi(c_1 + c_2)] + \exp[-\pi(d_1 + d_2)]\}$$

$$M = \exp[-\pi(c_1 + d_1 + c_2 + d_2)] \tag{15b}$$

CASE III. *One pair of complex conjugate, and one pair of real eigenvalues*:

$$p_1 = -a_1 + b_1i, \qquad \bar{p}_1 = -a_1 - b_1i, \qquad p_2 = -c_2, \qquad \bar{p}_2 = -d_2$$

$$\begin{aligned} b_1(d_2 - c_2)N &= (\exp[-\pi a_1])\{[(d_2 - a_1)(a_1 - c_2) - b_1^2][\exp(-\pi d_2) - \exp(-\pi c_2)]\sin b_1\pi \\ &+ b_1(c_2 - d_2)[\exp(-\pi d_2) + \exp(-\pi c_2)]\cos b_1\pi\} \end{aligned} \tag{16a}$$

$$M = \exp[-\pi(2a_1 + c_2 + d_2)] \tag{16b}$$

For any given double-pulse system, the stability characteristics can be quickly and straight-forwardly determined by calculating the p_i and $\bar{p}_i$ from Eqs. (9), evaluating N and M by means of the appropriate pair of expressions from the set (14a, b), (15a, b) and (16a, b), and then obtaining the two values of σ from Eq. (13). It should be noted that these calculations will always involve only real quantities.

From Theorem I it follows that for stability of a double-pulse system it is necessary that $(f_1 + f_2) \geq 0$. According to Eqs. (9), this is equivalent to the following theorem:[10]

THEOREM II: *If a double-pulse system is characterized in one half-period by eigenvalues p_1 and $\bar{p}_1$ and in the other half-period by eigenvalues p_2 and $\bar{p}_2$, then a necessary condition that the system be stable is that the arithmetic mean of the real parts of p_1, $\bar{p}_1$, p_2 and $\bar{p}_2$ be negative or zero. Alternatively, a sufficient condition that the system be unstable is that the arithmetic mean of the real parts of p_1, $\bar{p}_1$, p_2 and $\bar{p}_2$ be positive.*

By virtue of Theorem I and of relations of the form (9) it can be quite similarly proven that Theorem II is valid more generally for an n-pulse system. Moreover, it similarly follows that if during any period of an n-pulse system, $f(t)$ assumes constant values $f_1, f_2, \cdots, f_n$ during *unequal* time-intervals, then a necessary condition for stability is that the *weighted* mean of the real parts of the complex frequencies p_1 to p_n be negative or zero, the weights being proportional to the time-intervals corresponding to each of the p's.

It may be noted that the above necessary condition for the stability of a double-pulse system is in a sense less stringent than that of a system with constant elastic and damping coefficients, since in the latter case it would be required that each damping factor be positive, and not merely the average of the damping factors. It must be observed, on the other hand, that the theorems derived here give only necessary, but not sufficient, conditions for stability. This follows essentially from the fact that although the conditions $| \sigma_1 | \leq 1$, $| \sigma_2 | \leq 1$ always imply $\sigma_1 \sigma_2 \leq 1$, the converse is not necessarily true. As a consequence, a double-pulse system may be unstable even under conditions for which a constant-coefficient system would be stable. For example, it is well known that in the absence of damping forces a constant-coefficient system will be neutrally stable when the restoring force is positive ($k > 0$), leading to purely imaginary complex frequencies. However, as can be verified by putting $a_1 = a_2 = 0$ in Eqs. (14a, b) and noting that values of b_1 and b_2 (e.g. $b_1 = 1.5$, $b_2 = 0.5$) exist for which $| N | > 2$ and hence $| \sigma | > 1$, a double-pulse system may in such a case be unstable.

From Eqs. (14) and (16) in conjunction with (13) it is seen that although the necessary condition for stability stated here involves only the damping factors, the necessary *and* sufficient condition involves the natural frequencies as well. This is partly in contrast to a system with constant damping and spring characteristics, where the stability depends exclusively on the damping factors. Considering, for example, Case I, which is in practice the most important case, it can be seen from Eqs. (14a, b) that a large positive value of $(a_1 + a_2)$ tends to keep the values of $| N |$ and $| M |$ far below unity, and hence strongly tends to make the system stable. This result appears quite reasonable on physical grounds. Nevertheless, it will be found that for any given value of $(a_1 + a_2)$, i.e. for any given amount of damping in the average, there will exist values of the fre-

[10]The theorem which follows could also be directly derived by noting that for stability it is necessary that $| M | \leq 1$, and by using the expressions (14b), (15b) and (16b) for M.

quency ratio b_1/b_2 for which the system will be unstable. This can be verified by observing that for any given positive value of $(a_1 + a_2)$, however large, sufficiently large values of b_1/b_2 (or b_2/b_1) can be found for which $|N|$ as given by Eq. (14a) will be large enough to make at least one of the roots in Eq. (13) greater than unity in absolute value.

Examples

1. As an example illustrating the insufficiency of the necessary condition derived here for stability, suppose that in Case I $a_1 = 5$, $a_2 = -4$ and $b_1 = b_2 = 1.5$. Then the average of the damping factors a_1 and a_2 is here positive; nevertheless this system is unstable, for Eqs. (14a) and (14b) yield: $N = 38.0 \exp(-\pi)$, $M = \exp(-2\pi)$. The larger of the roots of Eq. (13) will be $|\sigma| = 1.64$, which exceeds 1 and hence implies instability.

2. The purpose of the second example is to show that a system pertaining to Case II can, contrary to systems with constant characteristics, be stable even when the average of a single pair of damping factors over a period, e.g. $(c_1 + d_2)/2$ or $(c_1 + c_2)/2$, is negative. Thus, let $c_1 = 1$, $c_2 = -1.333$, $d_1 = 1.333$ and $d_2 = 0.880$. Then $(c_1 + c_2) < 0$. Eqs. (15a, b) however give: $N = -0.209$, $M = 0.00169$. The roots (13) for σ are $\sigma_1 = 0.201$ and $\sigma_2 = 0.0081$, and the system is therefore stable.

It may be noted that in either of the above two numerical examples, it might have been difficult to predict in advance the stability or instability of the given system.

Conclusions

1. A necessary condition for the stability of a linear dynamic system with damping coefficient $f(t)$ and elastic coefficient both varying periodically with the period 2π is

$$\int_0^{2\pi} f(t)\, dt \geq 0.$$

2. In particular, a necessary condition for the stability of an n-pulse system is that the weighted arithmetic mean of the real parts of the $2n$ "complex frequencies" characterizing the system during any period of the variation of the damping and elastic coefficients be negative or zero. The weights must be proportional to the time-intervals, of any period, corresponding to each of the complex frequencies. For equal time-intervals, of course, the simple arithmetic mean is taken. The case $n = 2$ with equal time-intervals ("double-pulse"), was of especial concern here.

3. Contrary to systems with constant damping and elastic coefficients, the necessary and sufficient condition for the stability of a double-pulse system depends, in general, not only on the damping factors, or (negative) real parts of the complex frequencies, but also on the natural frequencies, or imaginary parts of the complex frequencies. Moreover, contrary to a system with constant coefficients, a double-pulse system can be stable even when not each of the damping factors is positive or zero.

4. The stability characteristics of any given double-pulse system with (positive and/or negative) damping can in practice be quickly calculated from the equations developed here.

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THE RANDOM VIBRATIONS OF A STRING*

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1. Introduction. In the theory of linear electrical networks one encounters systems of equations having the form

$$\sum_{k=1}^{n} (L_{jk}q_k'' + R_{jk}q_k' + G_{jk}q_k) = \sum_{k=1}^{n} E_{jk}\,, \qquad j = 1, \cdots, n \tag{1}$$

where the q_k represent charges, L_{jk} inductances, R_{jk} resistances, G_{jk} reciprocals of capacitance, the E_{jk} are random e.m.f's**, and primes denote differentiations with respect to time.

The theory of such systems of equations, quite carefully examined during the war years in applications to noise in electrical networks, turns out also to be applicable to the various mechanical systems—in particular, it is applicable to the system of a vibrating string with fixed end points. A general theory for the system of equations (1) has been developed by Uhlenbeck and Wang [2]. Some of the results which we shall obtain in this article have been derived without making use of the general theory. We shall make comparison with these results and in addition we shall derive several more results. The results of Uhlenbeck and certain of his co-authors [1, 5], have been derived directly from the differential equation of motion for the string. In order to apply the general theory to the vibrating string, it is necessary to discretize the string. We therefore assume that the string† is made up of $n + 2$ particles, (2 fixed, n vibrating) of equal mass m harmonically bound together by means of massless springs. Furthermore let us assume that this system of particles has random forces acting on it and that as a result the system vibrates, the vibration taking place in a plane. As a last assumption we suppose that the vibration takes place in a viscous medium so that each of the particles undergoes a damped vibration.

When we have obtained our results for the discretized system, we can derive the results for the case of a continuous string by a limiting procedure, namely that of letting n the number of vibrating particles go to infinity while the total mass and length of the system remains constant.

In this article we shall derive the following:

1) the characteristic function of the sum of the squares of the deviations of the displacement of the particles from their given initial positions—also the corresponding characteristic function for the continuous string,

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**The word random used here in a heuristic sense obviously needs more careful definition and interpretation. A more precise investigation than can be given here is found in an article by Uhlenbeck and Wang (see [2]).

†We adhere to the assumptions made in the usual derivation of the differential equation of the string, i.e. small vibrations, constant tension throughout, etc.

2) the mean square deviation of the displacements of the n particles from some given initial distribution and the corresponding mean square deviation for the string,

3) the mean square deviation of a single particle of the system given that initially it had a certain displacement θ, and the corresponding limiting case.

2. Certain preliminary considerations. Let the displacement of the k-th particle of the system be denoted by y_k . Let $F_k(t)$ denote the random force acting on the k-th particle. By a simple application of Newton's law of mechanics we have the result

$$my_k'' + Ky_k' - \frac{\tau}{d}(y_{k+1} - 2y_k + y_{k-1}) = F_k(t), \tag{2}$$

where K is a quantity dependent on the damping and τ is the tension in the springs connecting the particles. The tension is assumed to be constant at all times. The quantity d is the distance between particles, also assumed constant.

In order to prepare ourselves for the limiting case of the string, it will be necessary to define a density μ in such a way that $\mu d = m$. Assuming the case of $n + 2$ particles (n vibrating, 2 fixed) it is easy to see that the following relations will hold

$$d = \frac{L}{n+1}, \qquad m = \frac{M}{n+1}, \tag{3}$$

$$\mu = \frac{m}{d} = \frac{M}{L},$$

where M is the total mass of the system and L is the total length for the limiting case of the string. Both M and L are given constants. Also, let $K = Bd$ where B is a constant dependent on the magnitude of the damping which is dependent on the viscosity of the medium in which vibration is taking place.

With the use of these relations (2) becomes

$$\mu d y_k'' + B d y_k' - \frac{\tau}{d}(y_{k+1} - 2y_k + y_{k-1}) = F_k(t). \tag{4}$$

This equation is of the form (1) if the following relations hold:

$$L_{jk} = \delta_{jk}\mu d, \qquad R_{jk} = \delta_{jk}Bd,$$

$$G_{jj} = 2r^2, \qquad G_{j(j+1)} = G_{J(J-1)} = -r^2, \tag{5}$$

where δ_{jk} is the Kronecker delta, equal to one if $j = k$ and zero otherwise, and $r^2 = \tau/d$.

From physical considerations it is evident that the quantity $B = \beta/\mu$ where β is the actual damping coefficient for the medium in which vibration is taking place.

As Uhlenbeck and Wang [1] have pointed out in their discussion of the system (1) the $2n$ variables $[q_1(t), q_2(t), \cdots, q_n(t); q_1'(t), q_2'(t), \cdots, q_n'(t)]$ or in our case the collection $[y_1(t), \cdots, y_n(t); y_1'(t), \cdots, y_n'(t)]$ will form a $2n$-dimensional Markoff process governed by the Fokker-Planck or generalized diffusion equation

$$\frac{\partial P}{\partial t} = -\sum_{j=1}^{2n} \frac{\partial}{\partial x_j}(A_j P) + \frac{1}{2}\sum_{j,k=1}^{2n} \frac{\partial^2}{\partial x_j\, \partial x_k}(D_{jk}P), \tag{6}$$

where P is the probability density associated with the distribution of the quantities

$[x_1, x_2, \cdots, x_{2n}]$ Here $x_1, x_2, \cdots, x_{2n}$ are the $2n$ variables describing the collection $[y_1(t), \cdots, y_n(t); y_1'(t), \cdots, y_n'(t)]$. As derived in the article

$$A_j = \sum_{k=1}^{2n} a_{jk} x_k , \tag{7}$$

where the a_{jk} are the elements obtained from the matrix

$$A = \begin{bmatrix} 0 & I \\ -L^{-1}G & -L^{-1}R \end{bmatrix}, \tag{8}$$

and the D_{jk} are obtained from the matrix*

$$D = \begin{bmatrix} 0 & 0 \\ 0 & 2\kappa T L^{-1} R L^{-1} \end{bmatrix} \tag{9}$$

In the matrices (8) and (9), L, R and G are themselves matrices consisting of the elements L_{jk} . R_{jk} and G_{jk} first encountered in the system of equations (1) and which for our purposes may be defined by (5).

In order to solve the Fokker-Planck equation it turns out to be convenient to make the linear transformation

$$z_j = \sum_{k=1}^{2n} c_{jk} x_k , \tag{10}$$

and it then follows that (6) becomes

$$\frac{\partial P}{\partial t} = -\sum_{i=1}^{2n} \lambda_i \frac{\partial}{\partial z_i}(z_i P) + \frac{1}{2} \sum_{j,k=1}^{2n} \sigma_{jk} \frac{\partial^2 P}{\partial z_j \, \partial z_k}, \tag{11}$$

where the σ_{jk} are obtained from the matrix $CDC^{\sim}$, C being the matrix which diagonalizes the matrix A and which thereby yields the eigenvalues λ_j $(j = 1, 2, \cdots, 2n)$.

It is easy to show that the eigenvalues satisfy the determinantal equation

$$\begin{vmatrix} Q & P & 0 & \cdots & 0 & 0 \\ P & Q & P & \cdots & 0 & 0 \\ 0 & P & Q & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & P & Q \end{vmatrix} = 0, \tag{12}$$

$$Q \equiv \lambda^2 + \frac{B\lambda}{\mu} + \frac{2r^2}{\mu d}, \qquad P \equiv -\frac{r^2}{\mu d}$$

If we denote the determinant (12) by A_n we may show that the following difference equation is satisfied

*Here κ is Boltzmann's constant and T is the temperature of the medium in which the vibration is taking place.

$$A_n = QA_{n-1} - P^2A_{n-2} \tag{13}$$

subject to the boundary conditions $A_0 = 1$, $A_1 = Q$. Solving the difference equation by the usual methods we obtain

$$A_n = \frac{1}{(Q^2 - 4P^2)^{1/2}}\left[\left\{\frac{Q + (Q^2 - 4P^2)^{1/2}}{2}\right\}^{n+1} - \left\{\frac{Q - (Q^2 - 4P^2)^{1/2}}{2}\right\}^{n+1}\right] \tag{14}$$

yielding the evaluation of the determinant (12).

Upon finding the conditions under which $A_n = 0$, we obtain the equation

$$\lambda^2 + \frac{B\lambda}{\mu} + \frac{2r^2}{\mu d} = \frac{2r^2}{\mu d}\cos\frac{m\pi}{n+1}, \tag{15}$$

and so the $2n$ eigenvalues λ_i are given by

$$\lambda = -\frac{B}{2\mu} \pm \left[\frac{B^2}{4\mu^2} - \frac{4\tau}{\mu L^2}(n+1)^2\sin^2\frac{m\pi}{2n+2}\right]^{1/2} \qquad m = 1, \cdots, n \tag{16}$$

The evaluation of the matrix C which diagonalizes A is a little too long to present here but it is straight-forward. It turns out that the matrix C is given by*

$$\left.\begin{aligned} C_{p_{(m)},\,k} &= (\lambda_{p_{(m)}} + \beta)\sin\frac{km\pi}{n+1} \\ C_{p_{(m)},\,n+k} &= \sin\frac{km\pi}{n+1} \end{aligned}\right\} \begin{aligned} &k = 1, \cdots, n, \\ &m = 1, \cdots, n, \\ &p_{(m)} = 1, \cdots, 2n. \end{aligned} \tag{17}$$

After obtaining the matrix C, a simple evaluation of $CDC^{\sim}$, where D is given in (9), yields the σ_{jk} of the Fokker-Planck equation (11). This evaluation gives

$$\sigma_{jk} = \frac{2\kappa T\beta}{\mu d}\left(\sum_{m=1}^{n}\sin\frac{mj\pi}{n+1}\sin\frac{mk\pi}{n+1}\right), \tag{18}$$

where $\sigma_{jk} = \sigma_{j+1k}$ if j is odd and $\sigma_{jk} = \sigma_{jk+1}$ if k is odd. Evaluation of the sum in (18) shows that the σ_{jk} are given by the elements in

$$\begin{Vmatrix} \alpha & \alpha & 0 & 0 & \cdots & 0 \\ \alpha & \alpha & 0 & 0 & \cdots & 0 \\ 0 & 0 & \alpha & \alpha & \cdots & 0 \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ 0 & 0 & & \cdots & \alpha & \alpha \\ 0 & 0 & & \cdots & \alpha & \alpha \end{Vmatrix}, \tag{19}$$

where $\alpha = \kappa T\beta(n+1)^2/\mu L$.

*The significance of the bracket in $p_{(m)}$ is that corresponding to the value 1 for m we have $p_{(m)}$ equal to 1 and also 2, corresponding to $m = 2$ we have $p_{(m)}$ equal to 3 and also 4 etc.

The transformed Fokker-Planck equation (11) has for its solution a $2n$-dimensional Gaussian distribution with the average values

$$\bar{z}_i = b_{i0} \exp(\lambda_i t) \tag{20}$$

and covariance

$$\langle (z_i - \bar{z}_i)(z_k - \bar{z}_k) \rangle_{\text{Average}} = \frac{-\sigma_{ik}}{\lambda_i + \lambda_k} [1 - \exp\{(\lambda_i + \lambda_k)t\}] \tag{21}$$

where b_{i0} are the initial values of the z_i .

3. The characteristic function. Using the results obtained thus far we may now write the probability density function for the system of the n vibrating particles using a well-known result from the theory of multivariate Gaussian distributions. The probability density has the form

$$P(z_1, z_2, \cdots, z_{2n}) = R_n \exp\left\{-\frac{1}{2} \sum_{i,k} B_{ik}(z_i - \bar{z}_i)(z_k - \bar{z}_k)\right\}, \tag{22}$$

where the B_{ik} and R_n are constants determined from (20) and (21), and the values of λ are determined from (16).

We are now ready to obtain the characteristic function of the sum

$$\frac{1}{n+1} \sum_{i=1}^{n} (x_i - s_i)^2, \tag{23}$$

i.e. the characteristic function of the normalized* sum of the squares of the deviations of the displacements from some arbitrary initial values. This characteristic function, once we have obtained it, will at least in principle, enable us to obtain the characteristic function of

$$\frac{1}{L} \int_0^L [X(x, t) - S(x)]^2 \, dx \tag{24}$$

which is the corresponding limiting case of the finite sum (23). Once we have this we are theoretically able, by means of the Fourier Inversion Theorem of Probability, to obtain

$$Pr\left\{\frac{1}{L} \int_0^L [X(x, t) - S(x)]^2 \, dx < \epsilon\right\} \tag{25}$$

that is the probability that the mean square deviation of the string from a given initial distribution be less than some given quantity ϵ.

The required characteristic function is defined by

$$F_n(\xi) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} R_n \exp\left[\frac{i\xi}{n+1} \sum_{i=1}^{n} (x_i - s_i)^2 - \frac{1}{2} \sum_{i,k} B_{ik}(z_i - \bar{z}_i)(z_k - \bar{z}_k)\right] dV \tag{26}$$

*This normalization is necessary if our results for the limiting case of the continuous string are to be meaningful.

where the integration element $dV = dz_1 \cdots dz_{2n}$. If the integration in (26) is performed, we obtain the result

$$\left\{\exp\left[2i\xi \sum_{k=1}^{n} U_k^2 V_k\right]\right\} \prod_{k=1}^{n} V_k , \tag{27}$$

where

$$U_k = \frac{1}{n+1} \sum_{m=1}^{n} (\bar{x}_m - s_m) \sin \frac{mk\pi}{n+1}, \tag{28}$$

and

$$V_k = [1 - P_{2k-1,2k}]^{-1} \tag{29}$$

Here, for example,

$$P_{12} = \frac{4i\xi\{B_{11} + 2B_{12} + B_{22}\}}{(\lambda_1 - \lambda_2)^2(n+1)^2(B_{11}B_{22} - B_{12}^2)} \tag{30}$$

with similar definitions for P_{34}, and in general for $P_{2k-1,2k}$. The B_{jk} are the coefficients in the probability density. If we make use of the values of B_{jk} in terms of the means and covariances (20) and (21) we may write the P's in convenient form as

$$P_{12} = \frac{4i\xi}{(\lambda_1 - \lambda_2)^2} \int_0^t (e^{\lambda_1 t} - e^{\lambda_2 t})^2 \, dt \tag{31}$$

with similar definitions for $P_{2k-1,2k}$.

It can be shown rigorously that the characteristic function (26) has a limit as $n \to \infty$ and that the value of this limit is

$$\left\{\exp\left[2i\xi \sum_{k=1}^{\infty} u_k^2 v_k\right]\right\} \prod_{k=1}^{\infty} v_k \tag{32}$$

where

$$u_k = \frac{1}{L} \int_0^L [A(x, t) - S(x)] \sin \frac{k\pi x}{L} \, dx \tag{33}$$

and

$$\lim_{n\to\infty} \prod_{k=1}^{n} V_k = \prod_{k=1}^{\infty} v_k \tag{34}$$

The details of the limiting procedure are not as obvious as they might appear on first sight since the quantity n occurs in each term of the series and products of (27).

4. Derivation of averages. In this section we shall be interested in obtaining certain averages indicating the behavior of our system of n harmonically bound particles. Once we have the averaged behavior of the finite number of particles, it is but a single step to the case of the average properties of the string.

We first derive the mean square deviation of the displacement of the k-th particle from some initial position, that is we wish

$$\langle x_k^2 \rangle_{\text{Average}} = \langle (x_k - \bar{x}_k)^2 \rangle_{\text{Average}} + \bar{x}_k^2 \tag{35}$$

From (10) and (17) we may obtain

$$\bar{x}_k = \frac{2}{n+1} \sum_{r=1}^{n} \left(\frac{\bar{z}_{2r-1} - \bar{z}_{2r}}{\lambda_{2r-1} - \lambda_{2r}} \right) \sin \frac{kr\pi}{n+1} \tag{36}$$

With the use of (36) we obtain

$$\left. \begin{aligned} \bar{x}_k^2 = \sum_{r,s=1}^{n} & (B_r(t)B_s(t)x_{r0}x_{s0} + e^{-\beta t/2}B_r(t)x_{r0}P_{s0} \sin \omega_s t) \\ + \sum_{r,s=1}^{n} & (\omega_r^{-1}e^{-\beta t/2}B_s(t)x_{s0}P_{r0} \sin \omega_r t + \omega_r^{-1}\omega_s^{-1}e^{-\beta t}P_{r0}P_{s0} \sin \omega_r t \sin \omega_s t) \end{aligned} \right\} \tag{37}$$

where

$$B_r(t) \equiv e^{-\beta t/2} \left(\cos \omega_r t + \frac{\beta}{2\omega_r} \sin \omega_r t \right) \tag{38}$$

and

$$\omega_r \equiv \left\{ \frac{4\tau(n+1)^2}{\mu L^2} \sin^2 \frac{r\pi}{2n+2} - \frac{\beta^2}{4} \right\}^{1/2} \tag{39}$$

Similarly we may derive

$$\langle (x_k - \bar{x}_k)^2 \rangle_{\text{Average}} = \frac{E\mu}{4La^2\beta} \sum_{r=1}^{n} \frac{\sin^2 kr\pi/(n+1)}{\sin^2 r\pi/(2n+2)} \{1 - U_r\} \tag{40}$$

where

$$U_r = \frac{\beta e^{-\beta t}}{4\omega_r^2} \left(2\omega_r \sin 2\omega_r t - \beta \cos 2\omega_r t + 16a^2 \sin^2 \frac{r\pi}{2n+2} \right) \tag{41}$$

and

$$E = \frac{2\beta\kappa T}{\mu}, \qquad a^2 = \frac{\tau(n+1)^2}{\mu L^2}. \tag{42}$$

Putting (37) and (40) in (35) we obtain the result $\langle x_k^2 \rangle_{\text{Average}}$ that is the mean square deviation for the system of n particles.

The quantities x_{r0}, P_{r0} represent initial values of the displacements and momenta of the n particles respectively. We now average over a canonical ensemble of systems having all possible initial displacements and momenta after a long time has elapsed. The probability density P, which is a function of time, as we have found it in (21) approaches a limiting value as $t \to \infty$ which we may call the stationary density. This probability density is given by

$$R \exp \left\{ -\frac{\beta}{E} (P_{10}^2 + \lambda_1\lambda_2 x_{10}^2 + \cdots) \right\} \tag{43}$$

Using this stationary density or as it is usually called the density corresponding to the canonical distribution, we may examine the following question.

Given that a certain particle starts with the displacement θ at time zero, what is the mean square displacement of this particle after time t?

It is seen that what we want is the evaluation

$$\frac{\int \cdots \int \langle x^2 \rangle_{\text{Average}} \exp\left\{-\frac{\beta}{E}(P_{10}^2 + \lambda_1\lambda_2 x_{10}^2 + \cdots)\right\} dV}{\int \cdots \int \exp\left\{-\frac{\beta}{E}(P_{10}^2 + \lambda_1\lambda_2 x_{10}^2 + \cdots)\right\} dV} \tag{44}$$

The element dV is taken to be the element of volume in phase space $dP_{10} \cdots dP_{n0}$ $dx_{10} \cdots dx_{n0}$. Our condition that the initial displacement be of magnitude θ puts no restriction on the integration over the P but does put a restriction on the integration over the X. The restriction is that*

$$\theta \leq \sum_{r=1}^{n} x_{r0} \sin \frac{kr\pi}{n+1} \leq \theta + \epsilon \tag{45}$$

We must thus evaluate

$$\frac{\int \cdots \int x_{r0}x_{s0} \exp\left\{-\frac{\beta}{E}\sum_{r=1}^{n} \lambda_{2r-1}\lambda_{2r}x_{r0}^2\right\} dX}{\int \cdots \int \exp\left\{-\frac{\beta}{E}\sum_{r=1}^{n} \lambda_{2r-1}\lambda_{2r}x_{r0}^2\right\} dX} \tag{46}$$

where dX denotes the volume element $dx_{10} \cdots dx_{n0}$. To evaluate this we use a theorem of Hadamard [6]. We find the result

$$\frac{E}{2\beta} F_k + e^{-\beta t}\left\{\theta^2 - \frac{E}{2\beta} F_k\right\}\left\{\frac{\beta}{2}\frac{G_k(t)}{F_k} + \frac{1}{F_k}\frac{d}{dt} G_k(t)\right\}^2 \tag{47}$$

where

$$F_k \equiv \frac{2}{L}\sum_{r=1}^{n} \frac{\sin^2 kr\pi/(n+1)}{\lambda_{2r-1}\lambda_{2r}} \tag{48}$$

and

$$G_k(t) \equiv \frac{2}{L}\sum_{r=1}^{n} \frac{\sin \omega_r t}{\lambda_{2r-1}\lambda_{2r}\omega_r} \sin^2 \frac{kr\pi}{n+1} . \tag{49}$$

We have found the results for the system of n particles but a rigorous mathematical discussion shows that the process of replacing each term by the limit as $n \to \infty$ and replacing the sum from 1 to n by the sum from 1 to ∞ is justifiable. The results thus obtained are the same as those obtained by Uhlenbeck and Van Lear [5] and Uhlenbeck and Ornstein [1] by entirely different methods. An advantage of the approach given here is that the results for the finite system as well as for the string are obtained.

One further simplification in the above results can be made by recognizing that F_k in (48) can be evaluated explicitly. We note that

$$\lambda_{2r-1}\lambda_{2r} = 4a^2 \sin^2 \frac{r\pi}{2n+2} \tag{50}$$

*We use this restriction rather than $\theta = \sum_{r=1}^{n} x_{r0} \sin kr\pi/(n+1)$ because if we use the latter (46) becomes an indeterminate of the form 0/0. Instead we therefore use (45) and then let $\epsilon \to 0$.

so that

$$F_k \equiv \frac{1}{2La^2} \sum_{r=1}^{n} \frac{\sin^2 kr\pi/(n+1)}{\sin^2 r\pi/(2n+2)} . \tag{51}$$

An evaluation of this yields

$$F_k = \frac{k(n-k+1)}{La^2} \tag{52}$$

Thus the first term of (47) becomes

$$\frac{E}{2\beta} F_k = \frac{Ek(n-k+1)}{2La^2\beta} \tag{53}$$

and since

$$E = \frac{2\beta\kappa T}{\mu} , \tag{54}$$

we obtain in the limit as $n \to \infty$ since $k/(n+1) = x/L$ the term

$$\frac{\kappa LT}{\tau}\left(\frac{x}{L} - \frac{x^2}{L^2}\right) \tag{55}$$

This appears also in the results of Uhlenbeck and Van Lear for the limiting case of the string.

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ON THE DISTRIBUTION OF ENERGY IN NOISE- AND SIGNAL-MODULATED WAVES
II. SIMULTANEOUS AMPLITUDE AND ANGLE MODULATION*

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1. Introduction. In Part I[1] we have determined the distribution of energy in a carrier which is amplitude-modulated by a band of normal random noise or by a noise and a signal, and in an earlier work[2] the corresponding problem for frequency- or phase-modulated waves was considered. With the help of these results we now examine the case of a carrier which is simultaneously amplitude- and angle-modulated by a pair of normal random noise waves. Here, either noise wave may be delayed by an amount t_ϕ with respect to the other, but coherence between the two modulations is assumed to exist.

The investigation of this problem is of considerable interest in the study of the noise output of typical magnetron tubes, on the basis of a macroscopic model that treats the magnetron generator as producing a high-frequency oscillation, which in turn is simultaneously amplitude- and phase- or frequency-modulated by a comparatively low-frequency band of normal random noise, itself generated in the oscillator due to the "shot effect" or so-called primary noise which is inherent in the structure of the electron beam moving under the influence of the applied electric and magnetic fields. The present problem also provides us with results useful in communication theory when the interference appears as an amplitude and frequency distortion, or when the modulating signal can be used as a simplified model of a speech signal.† The central feature of this problem, as distinguished from those examined previously in Part I, is the fact that the modulating noise disturbances are now correlated with each other. The result is a "scrambling" of the separate modulations, which in turn produces an *asymmetrical* intensity spectrum, unlike those obtained before. This is a significant phenomenon which is entirely absent in the simpler situations where there is no correlation. Symmetrical distributions can occur for selected values of the relative delay t_ϕ between the amplitude and angle-modulations. Moreover, even in extreme cases, the clipping due to over-modulation causes comparatively little spreading of the spectrum and spectral shape is mainly governed by the degree of angle-modulation. Special attention is given to the example of coherent modulation without over-modulation. Specifically, the modulated carrier is represented by

$$g(y) \equiv V(t) = A_0(t) \exp\,[i(\omega_0 t + \Psi(t))], \qquad (\omega_0 = 2\pi f_0). \tag{2.1}$$

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[1]D. Middleton, Quart. Appl. Math. 9, 337 (1952).

[2]D. Middleton, Technical Report No. 99, Cruft Laboratory, Harvard University (March 1, 1950).

†We assume for the moment that any background noise accompanying the modulated carrier (and hence uncorrelated with it) is quite negligible. It is not, however, difficult to include this additional effect in the analysis.

2. The correlation function. For the amplitude modulation $V_{AN}(t)$ one has from (2.7) of reference 1

$$A_0(t) = (-A_0/2\pi) \int_C z^{-2} \exp\,(iz[1 + kV_{AN}(t + t_\phi)])\, dz, \tag{2.2a}$$

and for the angle modulation one can write

$$\Psi_{\phi M}(t) = D_0 V_{FN}(t); \qquad \Psi_{FM}(t) = D_0 \int^t V_{FN}(t')\, dt'. \tag{2.2b}$$

As before,[1] k (≥ 0) is an amplitude-modulation index, A_0 is the peak amplitude of the unmodulated carrier, and D_0 is the r-m-s phase or (angular) frequency-deviation per unit r-m-s modulating voltage (or current), viz.*: $D_0^2 = \langle \Psi(t)^2 \rangle_{s\cdot av.} / \langle V_{FN}(t)^2 \rangle_{s\cdot av.} = \omega_d^2 / \langle V_{FN}(t)^2 \rangle_{s\cdot av.}$ or $D_0^2 = \theta_d^2 / \langle V_{FN}(t)^2 \rangle_{s\cdot av.}$ respectively. For the moment we distinguish between the two (real) noise waves $V_{AN}(t + t_\phi)$ and $V_{FN}(t)$, the former of which represents the disturbance producing the amplitude distortion while the latter causes the phase- or frequency-modulation. The effects of intervals of over-modulation, during which the oscillator is cut off, are accounted for in the general expression (2.2a), which gives $A_0[1 + kV_{AN}(t + t_\phi)]$ when kV_{AN} exceeds -1 and vanishes when $kV_{AN} \leq -1$. Here t_ϕ represents a phase lead ($t_\phi > 0$), or lag ($t_\phi < 0$), of the instantaneous amplitude modulation with respect to the phase- or frequency-modulation. Such a phase difference is included because in general the mechanism producing the coherent modulations may not act instantaneously: there may be a definite phase lag or lead of the one over the other. We assume in any case that we are dealing with proper modulations, i.e., those whose highest significant frequency components are much less than the carrier frequency f_0, and that a direct modulation rather than a mixing process is responsible for the perturbed carrier. [See section 2, Part I, for further comments on this point.]

The auto-correlation function of the modulated carrier is written*

$$\begin{aligned} R(t) &= (1/2)\, Re\{\langle V(t_0, t_\phi) V(t_0 + t, t_\phi)^* \rangle_{s.av.}\} \\ &= (A_0^2/2)\, Re\Big\{ \exp\,(-i\omega_0 t)(4\pi^2)^{-1} \int_C z^{-2} e^{iz}\, dz \int_C \xi^{-2} e^{i\xi}\, d\xi \\ &\qquad \cdot [\exp\,(ikV_1 z + ikV_2 \xi + i\Psi_1 - i\Psi_2)]_{stat.av.} \Big\}. \end{aligned} \tag{2.3}$$

Here V_1 ($\equiv V_{AN}(t_0 + t_\phi)$) is the noise amplitude-modulation at an initial time t_0 and V_2 ($\equiv V_{AN}(t_0 + t_\phi + t)$) is the same disturbance at a time t later; Ψ_1 and Ψ_2 are respectively $\Psi(t_0)$ and $\Psi(t_0 + t)$, the noise angle-modulation. The statistical average is performed over the four random quantities $V_1, \cdots \Psi_2$ as indicated. They are not generally independent random variables, since $V_{AN}(t + t_\phi)$ and $V_{FN}(t)$ are assumed to come from the same source and to contain in common a significant fraction of their total spectral distributions. We note that

*The symbol $\langle\ \rangle_{s.av.}$ denotes the statistical or ensemble average over the appropriate random variables, while $\langle\ \rangle_{av.}$ indicates the infinite time average, cf. ref. 1.

$$[\exp(ikV_1z + ikV_2\xi + i\Psi_1 - i\Psi_2)]_{\text{stat.av.}}$$

$$= F_2(\zeta_1, \zeta_2, \zeta_3, \zeta_4; t) = \text{characteristic function of the random variables } kV_1, kV_2, \Psi_1, \text{ and } \Psi_2, \text{ for which}$$

$$\zeta_1 = z, \quad \zeta_2 = \xi, \quad \zeta_3 = 1, \quad \zeta_4 = -1.$$

Since $V_1 \cdots \Psi_2$ are *normal* random variables we can write at once[3]

$$F_2(\zeta_1, \zeta_2, \zeta_3, \zeta_4; t) = \exp\left(-\frac{1}{2}\sum_i^4 \sum_l^4 \zeta_i \zeta_l \mu_{il}\right) = F_2(z, \xi, 1, -1; t), \tag{2.4}$$

$$\text{when} \quad \zeta_1 = z, \quad \zeta_2 = \xi, \quad \zeta_3 = 1, \quad \zeta_4 = -1,$$

where the sixteen variances $u_{il}(j,l = 1 \cdots 4)$ are

$$\begin{aligned}
&\mu_{11} = \mu_{22} = k^2\langle V_1^2\rangle_{\text{s.av.}} = k^2\langle V_2^2\rangle_{\text{s.av.}} = k^2\Psi_A(0);\\
&\mu_{12} = \mu_{21} = k^2\langle V_1V_2\rangle_{\text{s.av.}} = k^2\Psi_A(t);\\
&\mu_{33} = \mu_{44} = \langle\Psi_1^2\rangle_{\text{s.av.}} = \langle\Psi_2^2\rangle_{\text{s.av.}} = \langle\Psi^2\rangle_{\text{s.av.}};\\
&\mu_{34} = \mu_{43} = \langle -\Psi_1\Psi_2\rangle_{\text{s.av.}}; \qquad \mu_{13} = \mu_{31} = k\langle V_1\Psi_1\rangle_{\text{s.av.}};\\
&\mu_{14} = \mu_{41} = -k\langle V_1\Psi_2\rangle_{\text{s.av.}}; \qquad \mu_{23} = \mu_{32} = k\langle V_2\Psi_1\rangle_{\text{s.av.}};\\
&\mu_{24} = \mu_{42} = -k\langle V_2\Psi_2\rangle_{\text{s.av.}}.
\end{aligned} \tag{2.5}$$

The characteristic function (2.4) becomes accordingly

$$F_2(z, \xi; 1, -1; t) = \exp[-\tfrac{1}{2}k^2(z^2 + \xi^2)\psi_A(0) - k^2z\xi\psi_A(t)] \cdot \exp[-D_0^2\Omega(t)] \cdot \exp\{-kD_0[z\Lambda^{(+)}(t) + \xi\Lambda^{(-)}(t)]\}, \tag{2.6}$$

in which

$$\Lambda^{(+)}(t) \equiv \Lambda(t, t_\phi) = \langle V_1 \cdot (\Psi_1 - \Psi_2)\rangle_{\text{s.av.}}/D_0; \tag{2.6a}$$

$$\text{and} \quad \langle V_2 \cdot (\Psi_1 - \Psi_2)\rangle_{\text{s.av.}}/D_0 \equiv \Lambda^{(-)}(t) = -\Lambda(-t, t_\phi),$$

and $\Omega(t)$ is given by[2]

$$\Omega(t) = \int_0^\infty W_{\phi M}(f)(1 - \cos\omega t)\, df, \qquad \text{or} \tag{2.7a}$$

$$= \int_0^\infty W_{FM}(f)(1 - \cos\omega t)\, df/\omega^2 \tag{2.7b}$$

for phase- or frequency-modulation respectively; $\psi_A(t)$ is the auto-correlation function of the noise wave $V_{AN}(t)$ which produces the amplitude-modulation.

There remains now to evaluate the *cross-correlation function* $\Lambda(t, t_\phi)$, relating the two modulating waves. We consider first phase-modulation. As in previous cases we express the modulating waves in terms of their Fourier transforms $S_T(f)_A$ and $S_T(f)_P$

[3] D. Middleton, Quart. Appl. Math. 5, 445 (1948).

for the amplitude and angle modulations respectively. Replacing statistical averages by their equivalent time averages, in virtue of the assumed stationarity (and ergodicity) of the random processes, we have finally

$$\Lambda(t, t_\phi) = \int_{-\infty}^{\infty} \lim_{T\to\infty} T^{-1}\{S_T(f)_A S_T(-f)_F\} \exp(it_\phi\omega)(1 - e^{-i\omega t})\, df. \tag{2.8}$$

Writing the amplitude-spectral densities $S_T(f)_A$, etc. in terms of a modulus $|S_T(f)_A|$ $(= |S_T(-f)_A|)$ and a phase $\phi(f)_A$ $(= -\phi(-f)_A)$, with similar expressions for $S_T(f)_F$, since we are dealing with a real wave, and noting furthermore that the mean spectral intensities of $V_{AN}(t)$ and $V_{FN}(t)$ are defined by

$$W_{AN}(f) \equiv \lim_{T\to\infty} 2\,|S_T(f)_A|^2 T^{-1}; \qquad W_{FN}(f) \equiv \lim_{T\to\infty} 2\,|S_T(f)_F|^2 T^{-1},$$

we can write the cross-correlation function Λ as

$$\Lambda_{\phi M}(t, t_\phi) = \int_0^\infty [W_{AN}(f) W_{\phi N}(f)]^{1/2} \cdot\{\cos(\omega t + \phi_A - \phi_F) - \cos[\omega(t_\phi - t) + \phi_A - \phi_F]\}\, df, \tag{2.9}$$

since the spectral densities are even functions of frequency, while the phase factors ϕ_F and ϕ_A are odd-functions. In the case of frequency-modulation we replace $S_T(f')_F$ by $S_T(f')_F/i\omega'$ in the above, according to Eq. (2.2b), and obtain finally the cross-correlation function

$$\Lambda_{FM}(t, t_\phi) = -\int_0^\infty [W_{AN}(f) W_{FM}(f)]^{1/2} \cdot\{\sin(\omega t_\phi + \phi_A - \phi_F) - \sin(\omega(t_\phi - t) + \phi_A - \phi_F)\}\, df/\omega. \tag{2.10}$$

The cross-correlation function $\Lambda(t, t_\phi)$ defined above in (2.6a) has a number of important properties:

(*i*) We observe at once from (2.9) or (2.10) that if the two noise waves have no spectral region in common, i.e., $W_{AN}(f)$ and $W_{FN}(f)$ do not overlap, the cross-correlation function vanishes, even though the two disturbances arise from a common source. Thus the amplitude- and frequency-modulations are no longer in any way correlated. In general, however, $W_{AN}(f)$ and $W_{FN}(f)$ may have a significant proportion of their power distributed in the same spectral region; the precise amount is measured by a quantity which we shall call the *overlap- or coherence-function* $\chi(t_\phi)$, viz.:

$$\chi(t_\phi) \equiv \langle V_1\Psi_1\rangle_{\text{s.av.}}/D_0 (\equiv \langle V_2\Psi_2\rangle_{\text{s.av.}}/D_0)$$

$$= \int_0^\infty [W_{AN}(f) W_{\phi N}(f)]^{1/2} \cos(\omega t_\phi + \phi_A - \phi_F)\, df$$

$$= \Lambda_{\phi M}(\infty, t_\phi) \tag{2.11a}$$

$$= -\int_0^\infty [W_{AN}(f) W_{FN}(f)]^{1/2} \sin(\omega t_\phi + \phi_A - \phi_F)\, df/\omega$$

$$= \Lambda_{FM}(\infty, t_\phi) + A \tag{2.11b}$$

respectively for phase- or frequency-modulation; A, as we shall see presently, cf. (2.15), is a constant depending on the behavior of both spectral densities $W_{AN}(f)$ and $W_{FN}(f)$ as $f \to 0$.

(*ii*) When t approaches zero, $\Lambda(t, t_\phi)$ vanishes identically; this is a direct consequence of the definition (2.6a) of the cross-correlation function.

(*iii*) When the time lag (or lead) t_ϕ of the amplitude-modulation with respect to the angle-modulation becomes indefinitely great (i.e., when $t_\phi \to \pm\infty$) we observe in the case of phase-modulation that since

$$\lim_{t_\phi \to \pm\infty} [\cos(\omega t_\phi + \phi_A - \phi_F) - \cos(\omega t_\phi + \phi_A - \phi_F - \omega t)] \tag{2.12}$$

$$= \mp 2\pi\omega \sin\frac{\omega t}{2}\,\delta(\omega - 0) = \mp\omega\left(\sin\frac{\omega t}{2}\right)\cdot\delta(f - 0),$$

we have*

$$\lim_{t_\phi \to \pm\infty} \Lambda_{\phi M}(t, t_\phi) = \mp\lim_{f\to 0}\left\{\frac{1}{2}\,[W_{AN}(f)W_{\phi N}(f)]^{1/2}\omega\sin\frac{\omega t}{2}\right\}, \tag{2.13}$$

which usually vanishes, unless the spectral densities together approach zero as $0(\omega^{-n})$, $(n \geq 2)$. For a band of noise whose zero-frequency density is finite or zero, (2.13) always vanishes. In a similar fashion one finds that in frequency-modulation

$$\lim_{t_\phi \to \pm\infty} \Lambda_{FM}(t, t_\phi) = \pm\frac{1}{2}\int_0^\infty [W_{AN}(f)W_{FN}(f)]^{1/2}[\delta(f - 0) - \delta(f - 0)]\,df = 0 \tag{2.14}$$

identically and quite independently of the spectral distributions of the modulating noise waves in the neighborhood of $f = 0$. The vanishing of the cross-correlation function in the limit of infinite delay times is explained if we note that the original wave-forms are now so separated in time that even a partial coherence between the two waves is completely lost. This is not true however when the major portion of the noise waves' energy lies in the almost zero-frequency region, cf. (2.13), as then the waveform changes so slowly in the course of time that even an infinite delay does not destroy coherence entirely. This can also be seen directly from the overlap function $\chi(t_\phi)$ when $t_\phi \to \pm\infty$.

(*iv*) A similar argument may be applied when $|\,t\,| \to \infty$ to show that

$$\lim_{t\to\pm\infty} \Lambda_{\phi M}(t, t_\phi) = \chi_{\phi M}(t_\phi); \qquad \lim_{t\to\pm\infty} \Lambda_{FM}(t, t_\phi) = \chi_{FM}(t_\phi) \mp \frac{1}{4}\,W_{AN}(0)W_{FN}(0); \tag{2.15}$$

here the time-dependent part of the cross-correlation function may or may not vanish, depending again on how much of the wave's energy is located in the spectral region $(0, \epsilon)$, as $\epsilon \to 0$.

(*v*) If the amplitude- and angle-modulations have the same spectra, i.e., $W_{AN}(f) = W_{FN}(f)$; $\phi_A = \phi_F$, a simplifying assumption that is ordinarily not at all critical, the calculation of the auto-correlation function (2.3) and the corresponding mean intensity spectrum $W(f)$ is greatly reduced, since the cross-correaltion function Λ takes a comparatively simple form. Note also that when the delay t_ϕ vanishes, one has

$$\Lambda_{\phi M}(t, 0) = \Omega_{\phi M}(t), \qquad \text{cf.} \qquad (2.7a); \qquad \psi_A(t) = \Omega_{\phi M}(\infty) - \Omega_{\phi M}(t) \tag{2.16a}$$

*A rigorous treatment recognizes these results as a form of the Dirichlet conditions.[4]

[4] H. S. Carslaw, *Theory of Fourier's Series and Integrals*, 3rd rev. ed. (Dover, N. Y., 1949), pp. 92-94.

$$\Lambda_{FM}(t, 0) = \Omega_{FM}(t), \qquad \text{cf.} \qquad (2.7b); \quad \psi_A(t) = \ddot{\Omega}_{FM}(t). \tag{2.16b}$$

For example, consider the case of identical gaussian noise spectra, $W(f) = W_0 \exp [-\omega^2/\omega_b^2]$; one readily finds from (2.9)-(2.11) that

$$\Lambda_{\phi M}(t, t_\phi) = (W_0\omega_b/4\pi^{1/2})[\exp(-\omega_b^2 t_\phi^2/4) - \exp(-\omega_b^2(t - t_\phi)^2/4)]; \tag{2.17a}$$

$$\chi_{\phi M}(t_\phi) = (W_0\omega_b/4\pi^{1/2}) \exp(-\omega_b^2 t_\phi^2/4),$$

$$\Lambda_{FM}(t, t_\phi) = (W_0/4)[\Phi(\omega_b t_\phi/2) - \Phi(\omega_b[t_\phi - t]/2)];$$

$$\chi_{FM}(t_\phi) = (W_0/4)\Phi(\omega_b t_\phi/2), \tag{2.17b}$$

where $\Phi(x) = 2\pi^{-1/2} \int_0^x \exp(-y^2)\, dy$.

The complete auto-correlation function (2.3) for the noise-modulated carrier can now be written

$$R(t) = (A_0^2/2) \exp\{-D_0^2\Omega(t)\} \, Re\Big\{\exp(-i\omega_0 t)(4\pi^2)^{-1}$$

$$\cdot \int_C z^{-2}\, dz \exp[iz - k^2\psi_A z^2/2 - kD_0\Lambda^{(+)}(t)z] \tag{2.18a}$$

$$\cdot \int_C \xi^{-2}\, d\xi \exp[i\xi - kD_0\Lambda^{(-)}(t)\xi - k^2\psi_A\xi^2/2] \cdot \exp[-k^2\psi_A(t)\xi z]\Big\}$$

$$= (A_0^2/2) \exp\{-D_0^2\Omega(t)\} \sum_{n=0}^{\infty} [(-k^2\psi_A)^n/n!] r_0(t)_A^n$$

$$\cdot \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} ([-kD_0(\psi_A\psi_F)^{1/2}]^{j+l}/j!l!)\lambda^{(+)}(t)^j\lambda^{(-)}(t)^l \tag{2.18b}$$

$$\cdot Re\{\exp(-i\omega_0 t) h_{0,n+j} h_{0,n+l}\},$$

where

$$h_{0,q+n} = (2\pi)^{-1} \int_C \exp(iz - k^2\psi_A z^2/2) z^{q+n-2}\, dz,$$

and (2.18c)

$$\Lambda^{(*)}(t)/(\psi_A\psi_F)^{1/2} \equiv \lambda^{(*)}(t); \qquad \psi_{A,F}(t)/\psi_{A,F} = r_0(t)_{A,F}\,.$$

The amplitude functions $h_{0,q+n}$ are given explicitly by Eqs. (2.11-2.12) of reference 1. The spectrum is found in the usual way by taking the Fourier transform of $R(t)$.

3. Remarks on power and spectra. As before, the mean total power W_T is obtained if one sets $t = 0$ in (2.18b) above, namely in the auto-correlation function $R(t)$, and the mean carrier power W_{f_0} follows from (2.18b) also, as t is allowed to become infinite. In the former case one gets a single infinite series ($n \geq 0$, $l = j = 0$), while in the latter a double series development ($n = 0$, $j, l \geq 0$), is the result. W_T and W_{f_0} may, however, be obtained in closed form, if we follow the procedure of section 2, ref. 1. We need now the joint first-order probability density $W_1(y, x)$ for the two correlated, normal random

variables $y = kV_{AN}(t + t_\phi)$ and $x = D_0 V_{\phi N}(t)$ or $D_0 \int^t V_{FN}(t')\,dt'$. This may be obtained directly from the characteristic function (2.4) and (2.5) for the *second-order* distribution on setting $\zeta_2 = \zeta_4 = 0$, which gives the characteristic function $F_1(\zeta_1, \zeta_3)$.

By inversion one finds[3] that

$$W_1(y, x) = \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} F_1(\zeta_1, \zeta_3) \exp(-iy\zeta_1 - ix\zeta_3)\, d\zeta_1\, d\zeta_3$$

$$= (2\pi\Delta^{1/2})^{-1} \exp\left\{-\frac{1}{2\Delta}\left[D_0^2\Omega(\infty)y^2 + k^2\psi_A x^2 - 2kD_0\chi(t_\phi)xy\right]\right\}, \qquad (3.1)$$

$$\Delta = k^2\psi_A D_0^2\Omega(\infty) - (kD_0\chi)^2.$$

Expressing $V(t)$, (2.1), now in terms of the random variables y and x, namely,

$$V(t) = A_0(1 + y)\exp(ix + i\omega_0 t), \qquad (y \geq -1);\ = 0, \qquad (y \leq -1),$$

we have

$$W_r = R(0) = \frac{1}{2}\langle |\, A_0(t)e^{i\Psi(t)}\,|^2\rangle_{s.av.}$$

$$= (A_0^2/2)\cdot\int_{y=-1}^{\infty} dy\,(1 + y)^2 \int_{-\infty}^{\infty} e^{ix - ix} W_1(y, x)\, dx \qquad (3.2)$$

$$= (A_0^2/2)\int_{-1}^{\infty} (1 + y)^2 W_1(y)\, dy,$$

where $W_1(y)$ is a normal probability density for which $\bar{y} = 0$, $\overline{y^2} = \psi_A k^2$, showing that the mean total power is the same as if there were amplitude-modulation only. This is what we would expect, since angle-modulation merely redistributes the original wave's energy among the carrier and sidebands without changing the total power (see sec. 2 of ref. 2). On the other hand, the mean power W_{f_0} remaining in the carrier is influenced by the phase- or frequency-modulation, because of the coherence between modulations. We have

$$W_{f_0} = \frac{1}{2}\,|\,\langle A_0(t)e^{i\Psi(t)}\rangle_{s.av.}\,|^2 = (A_0^2/2)\left|\int_{-1}^{\infty}(1 + y)\,dy\int_{-\infty}^{\infty} e^{ix}W_1(y, x)\,dx\right|^2$$

$$= (A_0^2/2)\left|\int_{-1}^{\infty}(1 + y)\,dy\int_{-\infty}^{\infty}(2\pi)^{-1}F_1(\zeta_1, 1)\,d\zeta_1\right|^2 \qquad (3.3a)$$

$$= (A_0^2/2)\exp[-D_0^2\Omega(\infty)]\left|\frac{1}{\pi^{1/2}}\int_{-(2k^2\psi_A)^{-1/2}}^{\infty} dz\right.$$

$$\left.\cdot(1 + z[2k^2\psi_A]^{1/2})\exp[-(z - iD_0\chi/(2\psi_A)^{1/2})^2]\right|^2, \qquad (3.3b)$$

which may be integrated in straightforward fashion with the help of the MacLaurin series for $\phi^{(0)}(x \pm a)$. We find the mean carrier power to be finally

$$W_{f_0} = (A_0^2/2)\exp[-D_0^2\Omega(\infty)]\left| h_{0,0} + ikD_0\chi \mid h_{0,1} \mid + k\psi_A^{1/2}\sum_{n=2}^{\infty}\frac{1}{n!}[iD_0\chi/\psi_A^{1/2}]^n\phi^{(n-2)}([k^2\psi_A]^{-1/2})\right|^2, \tag{3.4}$$

representative values of which are shown in Fig. 3.1, along with curves for the mean continuum power $W_c = W_\tau - W_{f_0}$. (Typical curves for W_τ are given in Fig. 2.2 of ref. 1.)

It is immediately apparent from (3.3b) that if there is no angle- or amplitude-modulation (i.e., $D_0 \to 0$, or $k \to 0$) one obtains the previous result for W_{f_0}, namely Eqs. (2.21), ref. 2 or (2.16), ref. 1, with corresponding expressions for the mean continuum power W_c. Furthermore, if there is no coherence between the modulations (so that $\chi \to 0$), the resulting power is proportional to the product of the individual carrier powers for angle- and amplitude-modulation alone, viz.:

$$W_{f0} = (A_0^2/2)h_{0,0}^2\cdot\exp[-D_0^2\Omega(\infty)], \qquad (\chi \to 0). \tag{3.5}$$

We distinguish now between phase- and frequency-modulation, according to the remarks of sec. 2, ref. 2. Figure 3.1 shows only the case of phase-modulation by noise with a gaussian spectrum for which $W_{\phi M}(0)$ is finite,* cf. Eq. (2.35), ref. 2. Increasing θ_d $(= D_0^2\langle V(t)_N^2\rangle_{\text{a-av.}})$, either by lessening the rate of sweep or by increasing the r-m-s phase deviation D_0, decreases the amount of power remaining in the carrier and makes noticeably greater the amount of energy available in the continuum. However, because of coupling with the amplitude-modulation, the scale of this effect is determined by (a), the extent of the amplitude variation ($\backsim\psi_A k^2$), and (b), the amount of coherence ($\backsim\chi$), which depends jointly on the time-delay t_ϕ and on the magnitude of the angle-modulation ($\backsim D_0^2\psi_F$). For very heavy over-modulation ($k \to \infty$) Eq. (3.4) reduces to

$$W_{f_0} \backsimeq (A_0^2/2)\left| [k^2\psi_A/2\pi]^{1/2}[1 + \exp(D_0^2\chi^2/2\psi_A) - (2^{1/2}D_0\chi/\psi_A^{1/2})\cdot\int_0^{D_0\chi/(2\psi_A)^{1/2}}\exp(y^2)\,dy + \frac{iD_0\chi\pi^{1/2}}{(2\psi_A)^{1/2}}\right|^2 \cdot\exp[-D_0^2\Omega(\infty)], \qquad (k \to \infty); \tag{3.6}$$

The integral in (3.6) is tabulated on page 32 of Jahnke and Emde's Tables (Dover, 1943).[5] Integrating (3.3b) by parts gives the appropriate asymptotic development when $D_0\chi/(2\psi_A)^{1/2}$ is large, for any value of $k(\geq 0)$. We have finally†

$$W_{f_0} \backsimeq (A_0^2/2)(k^2\psi_A^3/2\pi D_0\chi^4)\cdot|(1 + 2i/kD_0\chi + \cdots)|^2 \cdot\exp\{-1/k^2\psi_A + D_0^2[\chi^2/\psi_A - \Omega(\infty)]\}, \qquad (D_0\chi/(2\psi_A)^{1/2} \gg 1). \tag{3.7}$$

We return to the expression (2.18b) for the auto-correlation function of the modulated carrier. Applying the theorem of Wiener and Khintchine [(1.6), reference 1], we

*Since $W_{FM}(0) > 0$ for the same type of spectrum, $\Omega_{FM}(\infty)$ becomes infinite in such a way that W_{f_0} above vanishes. Consequently, all the wave's energy is distributed in the continuum; therefore $W_\tau = W_c$ and W_τ are independent of χ and D_0.

[5]Jahnke and Emde, *Tables of Functions*, 4th rev. ed. (Dover, N. Y., 1945).

†k never appears to any lower power than k^{-1} in the series in Eq. (3.7).

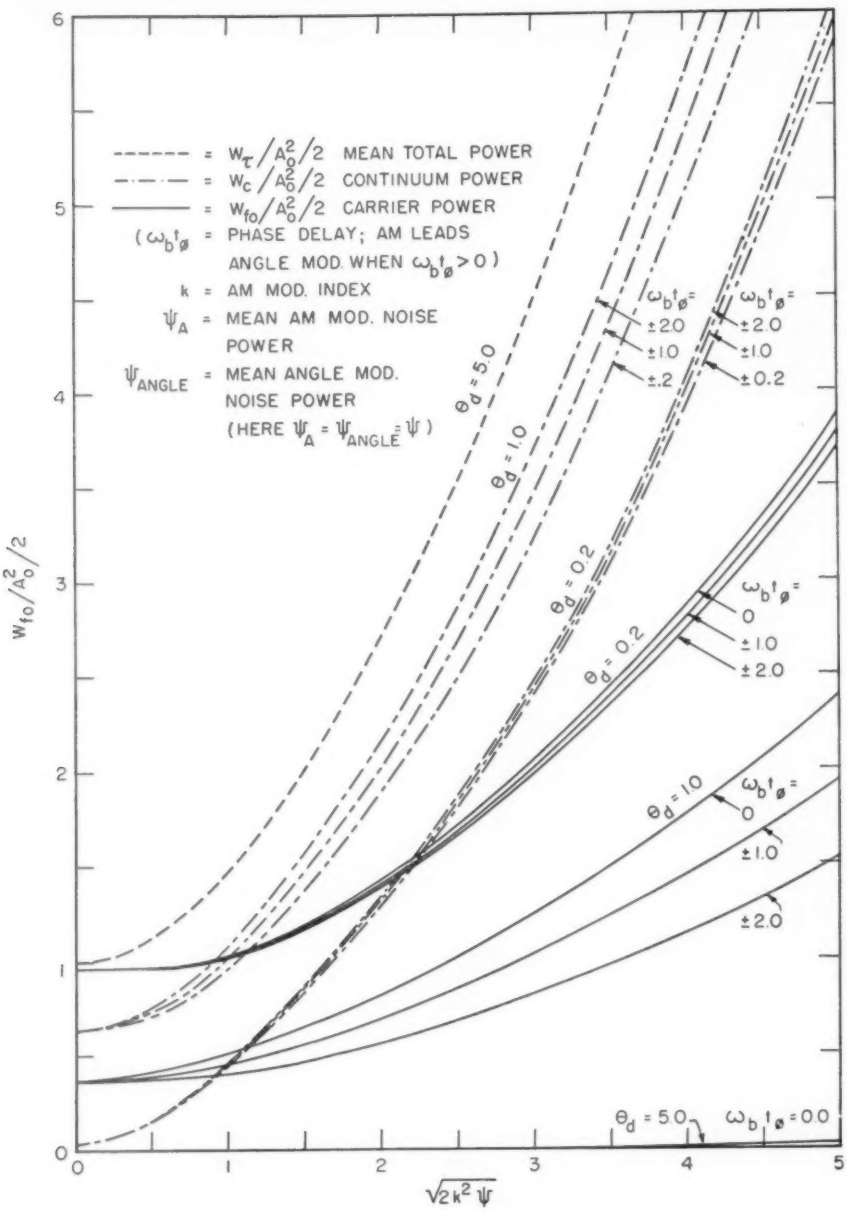

= $W_\tau/A_o^2/2$ MEAN TOTAL POWER
= $W_c/A_o^2/2$ CONTINUUM POWER
= $W_{fo}/A_o^2/2$ CARRIER POWER
($\omega_b t_\phi$ = PHASE DELAY; AM LEADS ANGLE MOD. WHEN $\omega_b t_\phi > 0$)
k = AM MOD. INDEX
ψ_A = MEAN AM MOD. NOISE POWER
ψ_{ANGLE} = MEAN ANGLE MOD. NOISE POWER
(HERE $\psi_A = \psi_{ANGLE} = \psi$)
$\theta_d = 5.0$
$\theta_d = 1.0$
$\theta_d = 0.2$
$\omega_b t_\phi$ = ±2.0 ±1.0 ±.2
$\omega_b t_\phi$ = ±2.0 ±1.0 ±0.2
$\omega_b t_\phi$ = 0 ±1.0 ±2.0
$\omega_b t_\phi$ = 0 ±1.0 ±2.0
$\theta_d = 5.0$ $\omega_b t_\phi = 0.0$
$W_{fo}/A_o^2/2$
$\sqrt{2k^2\psi}$
0 1 2 3 4 5 6

FIG. 3.1. Mean power in a carrier simultaneously amplitude and phase modulated by correlated random noise

distinguish three general types of spectral contribution: (i) "pure" angle-modulation of the carrier by noise, i.e., (*noise* × *carrier*), or $(n \times c)$ terms, for which $(n = j = l = 0)$; (ii) "pure" angle-modulation times "pure" amplitude-modulation harmonics, namely $(n \times c)_{\text{angle}} \cdot (n \times c)_{AM}$ components, for which in turn $(n \geq 1, j = l = 0)$; and finally (iii), the cross-modulation products $(n \times c)_{\text{angle}} \times (n \times c)_{AM}$, $(n \geq 0, j + l \geq 1)$, which arise from the coherence of the simultaneous angle- and amplitude-modulations. (We use the term "pure" above to indicate the independent or incoherent character of the modulation vis-à-vis the carrier and the other modulation.) Components of types (i) and (ii) are readily seen to yield only the usual symmetrical power spectra (about f_0). Modulation products belonging to group (iii), however, are more complex: these yield both symmetrical and *anti-symmetrical* distributions. As we shall see presently, this anti-symmetry is a direct consequence of the coherence between the two noise modulations and depends critically on the amount of delay t_ϕ between them.

We observe that the terms in (2.18b) for which $j + l =$ odd $(n \geq 1)$ are responsible for the spectral asymmetry. To see this we note first that terms involving $\lambda^{(+)}(t)$ and $\lambda^{(-)}(t)$ in the auto-correlation function $R(t)$, that are odd in t, yield spectral contributions which are odd (i.e., antisymmetrical) in $(f_0 - f)$. Furthermore, such terms have as a factor $\sin \omega_0 t$, so that $R(t)$ itself remains an even function, as required from the nature of $R(t)$. Now $\lambda^{(+)}$ and $\lambda^{(-)}$ may be split into two parts, one of which is even and the other odd, in such a way that if we let $\mu^{(+)}(t)^j \mu^{(-)}(t)^l / j!l!$ represent the summand of the double series in (2.18b), including the amplitude factors $h_{0,q}$, we may write symbolically $(E_j + 0_j)^j (-E_l + 0_l)^l / j!l!$, where E and 0 are respectively the even and odd parts of $\mu^{(m)}(t)$. Then by direct expansion one gets for $(j + l =$ odd$)$ only odd terms, and for $(j + l =$ even$)$, only even terms. Let us illustrate with the simplest, but most important case, for which $(j + l = 1, n = 0)$, and to facilitate the discussion without in any way changing its essential features, let us further require that $D_0 \psi_F^{1/2}$ be so small that we can set $\exp[-D_0^2 \Omega(t)]$ equal to unity. We have as before to distinguish between the generally dissimilar situations of phase- or frequency-modulation. From (2.18b), (2.9) and (2.10), we have for the part of the auto-correlation function for which $(j + l = 1, n = 0)$

$$R(t)_{n=0;j+l=1} \doteq (A_0^2/2) k D_0 (\psi_A \psi_F)^{1/2} [\lambda^{(+)}(t) + \lambda^{(-)}(t)] \, Re\{\exp(-i\omega_0 t) h_{0,0} \cdot h_{0,1}\} \tag{3.8a}$$

$$\begin{Bmatrix} \phi M \\ FM \end{Bmatrix} = A_0^2 k D_0 \sin \omega_0 t \{-i h_{0,0} \cdot h_{0,1}\} \Big[\int_0^\infty [W_{AN}(f') W_{(\phi N, FN)}(f')]^{1/2} \sin \omega' t \begin{Bmatrix} \sin(\omega' t_\phi + \phi_A' - \phi_F') \\ \cos(\omega' t_\phi + \phi_A' - \phi_F')/\omega' \end{Bmatrix} df' \Big], \tag{3.8b}$$

respectively for phase- or frequency-modulation. The power spectrum for these cross-products becomes

$$W(f)_{(\phi M, FM)} = A_0^2 k D_0 (-i h_{0,0} \cdot h_{0,1}) \cdot \int_0^\infty Q_{(\phi M, FM)}(f') [\delta(f' - [f_0 + f]) - \delta(f' - [-f_0 - f]) + \delta(f' - [f_0 - f]) - \delta(f' - [-f_0 + f])] \, df', \tag{3.9}$$

where

$$Q_{\phi M}(f') = [W_{AN}(f')W_{\phi N}(f')]^{1/2} \sin [\omega' t_\phi + \phi_A' - \phi_F'],$$

and

$$Q_{FM}(f') = [W_{AN}(f')W_{FN}(f')]^{1/2} \cos (\omega' t_\phi + \phi_A' - \phi_F')/\omega'. \tag{3.9a}$$

Since both f_0 and f are positive, for obvious physical reasons, it follows immediately that the spectral density (3.9) reduces to

$$W(f) = A_0^2 k D_0(-ih_{0,0}\cdot h_{0,1}) \begin{cases} Q(f_0 - f), & f_0 \geq f \geq 0 \\ -Q(f - f_0), & f \geq f_0 > 0 \end{cases}, \tag{3.10}$$

inasmuch as the integrals over Q in (3.9) are zero for all negative values of f_0 and f, and because $Q(f_0 + f)$ is vanishingly small, Q being a *low-frequency* spectral distribution of the type $W_0 \exp(-\omega^2/\omega_b^2)$. The asymmetrical nature of the spectrum is at once evident from (3.10). Actually, the discontinuity at $f = f_0$ does not exist, but is due here to our simplifying approximation that $\exp[-D_0^2\Omega(t)] \doteq 1$. Including the expo-

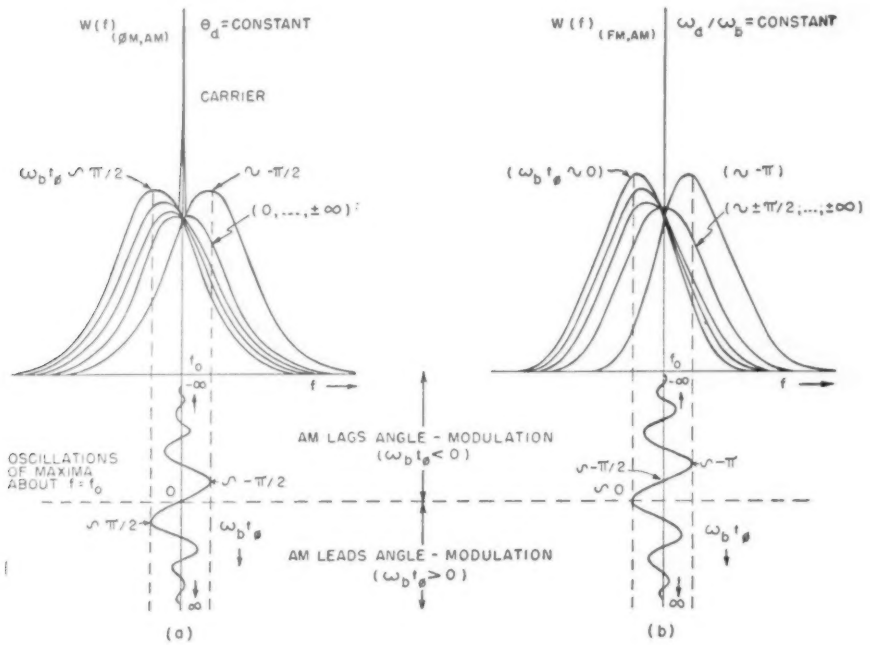


FIG. 3.2. Symmetrical and asymmetrical noise spectra for correlated noise modulation.

nential factor removes the discontinuity but does not alter the asymmetry; a specific example is considered in section 4(b) following. Similar remarks apply to the terms in $R(t)$, (2.18b), for which $n \geq 1$ and $l + j$ = odd ≥ 1.

Figure (3.2) shows typical spectra for a fixed amount of over-modulation and for various values of delay; (a) illustrates the case of phase-modulation by noise, while (b)

gives the corresponding case of frequency-modulation by the same noise wave. The noise causing the angle-modulation is here identical with that producing the amplitude-distortion of the carrier, but is delayed by t_ϕ seconds. We note at once that phase-modulation *leads* the frequency-modulation by approximately a quarter period. This is understandable if we remember that the frequency-modulation is the *integral* of the phase-modulation (for the same modulating wave), cf. (3.8), and consequently lags behind the latter in time by a delay $t_\phi \backsim \pi/2\omega_b$ on the average. In the case of simultaneous amplitude- and angle-modulation by a sine-wave alone this is exactly the delay. Now, as the delay or lead becomes progressively greater, the maxima of the spectral distributions, and therefore the distributions themselves, oscillate about f_0 in ever-decreasing swings until for infinite lags or leads all coherence between the modulations is lost, and only a symmetrical spectrum remains. The decay of the swings' maxima is quite rapid for phase-modulation and somewhat less so for frequency-modulation, cf. (2.17). The oscillations of the spectral maxima are also shown in Fig. (3.2) above; we remark, however, that values (of $\omega_b t_\phi$) at the points of symmetry ($f = f_0$) and maximum asymmetry are only approximate and do not represent a purely periodic phenomenon. The precise values depend in a complicated way on the spectral shape of the modulating noise, the amount of over-modulation, and the intensity of the angle-modulation. From this we see also that a symmetrical spectrum alone is not sufficient evidence upon which to postulate a lack of coherence between modulations. For a given level of clipping, the power (i.e. the area under the spectral curve) in the continuum is a constant, independent of the phase-differences between modulations, (but not necessarily independent of the type of angle-modulation).

As noted above, the energy spectrum may be resolved into three principal orders of modulation products (i)-(iii), viz.:

$$W(f) = W(f)_{(angle,\, n=0=j=l)} + W(f)_{(angle\cdot AM;\, n\geq 1,\, j=l=0)} + W(f)_{(angle\times AM;\, n\geq 0,\, j+l\geq 1)} = W(f)_{\mathrm{I}} + W(f)_{\mathrm{II}} + W(f)_{\mathrm{III}}, \tag{3.11}$$

respectively. When the deviation of the angle-modulation is very large compared to the rate of deviation, so that asymptotic developments are possible in the manner of section 2 (reference 2), we find from (2.18) that

$$W(f)_{\mathrm{I+II}} \backsimeq A_0^2 \sum_{n=0}^{\infty} \{(-k^2\psi_A)^n/n!\} h_{0,n}^2 \cdot \sum_{q=0}^{\infty} \int_0^{\infty} c^{(2q)}(t) r_0(t)_A^n \exp\left(-D_0^2\Omega^{(2)}t^2/2\right) \cos(\omega_0 - \omega)t\, dt, \tag{3.12}$$

where the polynomials $c^{(2q)}(t)$ yield the asymptotic development in descending powers of ω_d^2/ω_b^2 or θ_d^2; thus $c^{(2q)}(t)$ yields $(\omega_d/\omega_b)^{-2q}$ or θ_d^{-2q} multiplied by appropriate constants. Specifically we have

$$c^{(0)}(t) = 1; \qquad c^{(2)}(t) = \frac{-D_0^2\Omega^{(4)}}{4!}\, t^4; \qquad c^{(4)}(t) = \left(\frac{-D_0^2\Omega^{(6)}}{6!}\, t^6 + \frac{D_0^4\Omega^{(4)2}}{2!4!^2}\, t^8\right);$$

$$c^{(6)}(t) = \left(\frac{-D_0^2\Omega^{(8)}}{8!}\, t^8 + \frac{2}{2!6!4!}\, D_0^4\Omega^{(4)}\Omega^{(6)} t^{10} - \frac{D_0^6\Omega^{(4)3}}{3!4!^3}\, t^{12}\right);$$

$$c^{(8)}(t) = \left(\frac{-D_0^2\Omega^{(10)}}{10!}t^{10} + \frac{D_0^4}{2!}\left[\frac{\Omega^{(6)2}}{6!^2} + \frac{2\Omega^{(4)}\Omega^{(8)}}{8!4!}\right]t^{12} - \frac{3D_0^6\Omega^{(4)2}\Omega^{(6)}}{3!4!^2 6!}t^{14} + \frac{D_0^8\Omega^{(4)4}}{4!\cdot 4!^4}t^{16}\right); \quad \text{etc.} \tag{3.13}$$

where

$$\Omega_{FM}^{(2n)} = (-1)^{n-1}b_{2n-2}\,; \qquad n \geq 1; \qquad \Omega_{\phi M}^{(2n)} = b_{2n}(-1)^{n-1}; \qquad n \geq 0;$$

$$b_n = \int_0^\infty \omega^2 W(f)_{(FM,\phi M)}\, df.$$

The spectrum of the cross-modulation is more complicated, but can be handled in a similar fashion. One has then to consider terms of the following type:

$$(-1)\, Re\left\{h_{0,n+j}\cdot h_{0,n+j}\int_0^\infty r_0(t)^n c^{(2q)}(t)\Lambda(t,\, t_\phi)^j \Lambda(-t,\, t_\phi)^l \cdot \exp\left(-D_0^2\Omega^{(2)}t^2/2 - i\omega_0 t\right)\cos\omega t\, dt\right\}.$$

On the other hand, if the amplitude-modulation is excessive, so that over-modulation occurs approximately half the time, the asymmetrical components are suppressed, and one has essentially a symmetrical spectrum, which nevertheless is influenced by the coherence between the angle- and amplitude-modulations. The analytical explanation is immediate: only $h_{0,n+q}$ $(n + q = \text{even})$ does not vanish when $(2k^2\psi_A)^{1/2} \to \infty$, except $h_{0,1}$, which becomes independent of $(2k^2\psi_A)^{1/2}$, cf. Eqs. (2.12a-c), ref. 1. Consequently, $j + l$ is always even, and only symmetrical terms remain. Since these components are all of order $(k^2\psi_A)$, the contribution of the single remaining antisymmetrical pair $(j + l = 1,\ n = 0)$, which are $0(k\psi_A^{1/2})$, becomes ignorable in the limit $(2k^2\psi_A)^{1/2} \to \infty$. Physically, it is the amplitude distortion which effectively determines the "cut-off" and "cut-in" of the oscillator. The angle-modulation alone does not alter the amplitude, changing only the spacing between zero-crossings, and therefore one expects that the strong amplitude-modulation controls the duration of the "on"- and "off"-periods of the wave.

4. Coherent and incoherent modulations. Two important cases remain to be considered in some detail: *one*, the situation of no coherence $(\Lambda = 0)$ between modulations, and *two*, the case of coherence, but no over-modulation. A discussion of *one* and *two* follows below in parts (a) and (b).

(a) *No coherence*:

The general problem simplifies greatly when there is no coherence or coupling between the modulating waves, a condition represented analytically by the fact that $\Lambda(t,\, t_\phi)$ vanishes, and physically, by the complete independence of the noise waves. The auto-correlation function (2.18) reduces now to

$$R(t) = r_0(t)_{angle}\cdot R_0(t)_{AM}\cos\omega_0 t \tag{4.1}$$

$$= (A_0^2/2)\exp\left[-D_0^2\Omega(t)\right]\sum_{n=0}^{\infty}\{(-k^2\psi_A)^n/n!\}r_0(t)_A^n h_{0,n}^2\cos\omega_0 t,$$

and the intensity spectrum $W(f)$ $(= W(f)_{\mathrm{I}} + W(f)_{\mathrm{II}})$, cf. Eq. (3.11), contains no cross terms of the type $W(f)_{\mathrm{III}}$. The mean total power in the wave is given by Eq. (2.17), reference 1, as before; the power in the carrier and continuum is also easily found from (3.4).

Let us consider the specific example of phase-modulation, and let us assume a gaussian spectral distribution for the modulating noise; (in general, the two noise waves need not have identical spectra or powers.) The spectrum follows at once from the cosine Fourier transform of Eq. (4.1) and we obtain finally with the help of (c), section 2, reference 2,

$$W(f) = \frac{A_0^2 \pi^{1/2} \exp(-\theta_d^2)}{\omega_b} \left\{ h_{0,0}^2 \, \delta(f - f_0) + h_{0,0}^2 \sum_{q=1}^{\infty} \frac{\theta_d^{2q} \exp(-\beta^2/q)}{q! q^{1/2}} \right.$$
$$\left. + \sum_{q=0}^{\infty} \sum_{n=1}^{\infty} \frac{\theta_d^{2q}}{q!} \frac{(-k^2\psi_A)^n}{n!} h_{0,n}^2 \frac{\exp(-\beta^2/(q + n\omega_A^2/\omega_b^2))}{(q + n\omega_A^2/\omega_b^2)^{1/2}} \right\}, \qquad \beta = \frac{\omega_0 - \omega}{\omega_b}. \tag{4.2}$$

The first term represents the carrier's contribution, the second, the spectrum due to phase-modulation alone, and the final term, the distribution $W(f)_{\mathrm{II}}$, attributable to the noncoherent modulation products between the angle- and amplitude-modulations. When excessive over-modulation occurs, we find that

$$W(f) \backsimeq A_0^2 \frac{k^2\psi_A \exp(-\theta_d^2)}{2\pi^{1/2}\omega_b} \left\{ \delta(f - f_0) + \sum_{q=1}^{\infty} \frac{\theta_d^{2q}}{q!} \left(\left(\frac{\exp(-\beta^2/q)}{q^{1/2}} \right. \right. \right.$$
$$\left. + \frac{\pi \exp[-\beta^2/(q + \omega_A^2/\omega_b^2)]}{2(q + \omega_A^2/\omega_b^2)^{1/2}} \right)$$
$$+ \sum_{n=0}^{\infty} \left(\frac{(2n)! \exp[-\beta^2/(q + (n+2)\omega_A^2/\omega_b^2)]}{n!^2 2^{2n}(2n+2)(2n+1)[q + (n+2)\omega_A^2/\omega_b^2]^{1/2}} \right.$$
$$\left. \left. \left. + \frac{(2n)! \exp(-\beta^2/(n+2)\omega_A^2/\omega_b^2)}{n!^2 2^{2n}(2n+2)(2n+1)[(n+2)\omega_A^2/\omega_b^2]^{1/2}} \right) \right) \right\}, \qquad k^2 \to \infty, \tag{4.3}$$

and when there is ignorable over-modulation we have simply

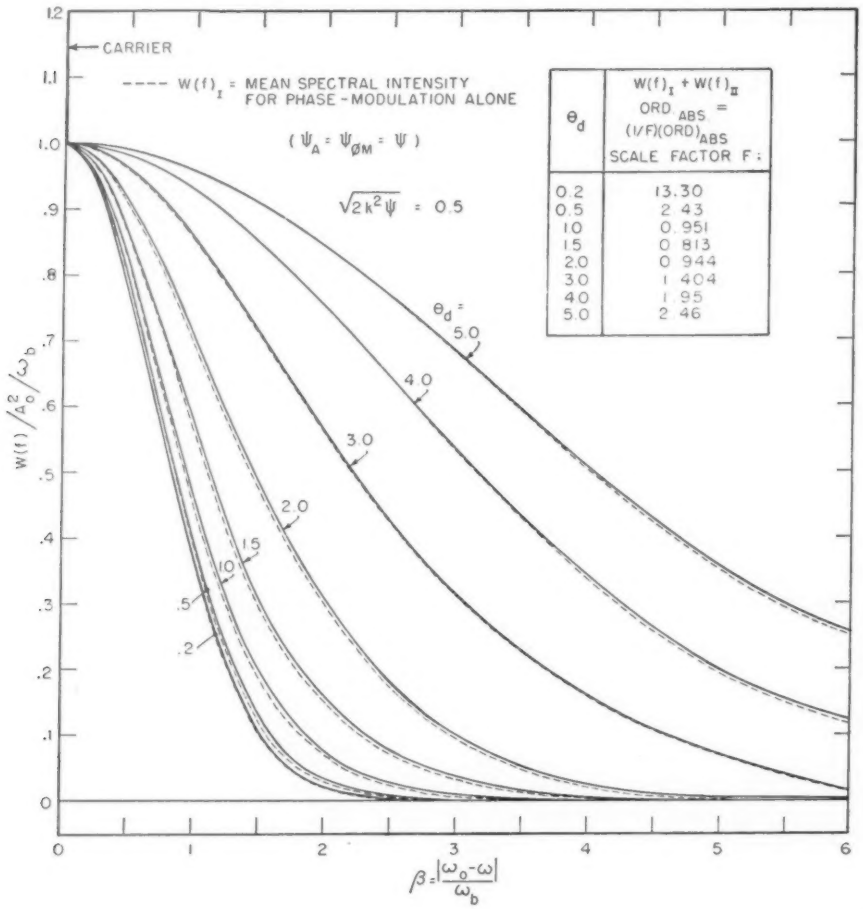
$$W(f) \doteq A_0^2 \frac{\pi^{1/2} \exp(-\theta_d^2)}{\omega_b} \left\{ \delta(f - f_0) + \sum_{q=1}^{\infty} \frac{\theta_d^{2q}}{q! q^{1/2}} \exp(-\beta^2/q) \right.$$
$$\left. + k^2\psi_A \sum_{q=0}^{\infty} \frac{\theta_d^{2q} \exp[-\beta^2/(q + \omega_A^2/\omega_b^2)]}{q!(q + \omega_A^2/\omega_b^2)^{1/2}}, \qquad (2k^2\psi_A)^{1/2} \leq 0.6. \right. \tag{4.4}$$

Figure (4.1) shows the energy spectrum for $(2k^2\psi)^{1/2} = 0.5$ and for various values of the deviation ratio θ_d . A general, but very slight broadening of the spectrum results from the additional modulation. Similar remarks apply for frequency-modulation (detailed calculations for which may be carried out along the lines of section 2(b), reference 2). In general, because the spread in the spectrum due to clipping is small, even in the extreme case of 50 per cent over-modulation, as can be seen from Fig. (2.1), reference 1, we expect little additional broadening of the spectrum due to this effect. For large values of θ_d^2 or ω_d^2/ω_b^2 the spectral intensity $W(f)$ is obtained in asymptotic form from

(3.12) and (3.13); in the instance of a gaussian spectrum for the amplitude-modulation ($\backsim W_A \exp(-\omega^2/\omega_A^2)$) we have

$$W(f) \backsimeq \sum_{n=0}^{\infty} \frac{(-k^2\psi_A)^n}{n!} h_{0,n}^2 \{\text{Eq. (2.33), ref. 2, modified}\}, \tag{4.5}$$

where now Eq. (2.33), reference 2 is easily modified by first multiplying numerator and denominator of each term which contains a factor $\phi^{(2q)}$ by a suitable power of D_0^2, so



θ_d	$W(f)_I + W(f)_{II}$ ORD. ABS. = (1/F)(ORD)$_{ABS}$ SCALE FACTOR F:
0.2	13.30
0.5	2.43
1.0	0.951
1.5	0.813
2.0	0.944
3.0	1.404
4.0	1.95
5.0	2.46

FIG. 4.1. Mean intensity spectrum for simultaneous uncorrelated amplitude and phase-modulation of a carrier by noise (no over-modulation).

that $[D_0^2\Omega^{(2)}]^q$ results in the denominator, and then replacing $[D_0^2\Omega^{(2)}]^q$ so obtained by $[D_0^2\Omega^{(2)} + \omega_A^2 n/2]^q$; the argument of the $\phi^{(2q)}$ is similarly changed from β'' to $\beta''/(1 + \omega_A^2 n/2D_0^2\Omega^{(2)})^{1/2}$.

(b) *Coherence, but no over-modulation*

This is a particularly important case because it is the simplest one which incorporates all the new features of coherence between modulations, and because it offers a promising macroscopic model for the behavior of noise in a magnetron, as described at the beginning of this section.

When over-modulation effects are ignorable (<1 per cent of the time), only the amplitude functions $h_{0,0}$ ($\doteq 1$) and $h_{0,1}$ ($\doteq i$) yield significant contributions, and we obtain from Eq. (2.18b) the following spectral components: $(\text{noise} \times \text{carrier})_{\text{angle mod.}}$, for which $(n = j = l = 0)$; $(n \times c)_{\text{angle}} \cdot (n \times c)_{AM}$, for which $(n = 1, j = l = 0)$; and finally $(n \times c)_{\text{angle}} \times (n \times c)_{AM}$, where $(n = 0, l + j = 1)$, all of which are represented in (4.6) below in the order listed:

$$R(t) \doteq (A_0^2/2) \exp [-D_0^2\Omega(t)]\Big\{\cos \omega_0 t + k^2\psi_A r_0(t)_A \cos \omega_0 t + 2kD_0\Big(\int_0^\infty Q(f') \sin \omega' t \, df'\Big) \sin \omega_0 t\Big\}, \qquad (2k^2\psi_A)^{1/2} \leq 0.6. \tag{4.6}$$

Here $Q(f')$ is given by Eq. (3.9a). We observe at once that the mean total power is the same as if there were no angle-modulation at all, which is not surprising, as angle-modulation alone does not change the energy content of a modulated carrier. Note also that the cross-correlation term never contributes to the total power (by definition, cf. (2.6a)), but represents a redistribution of the existing energy, determined by the carrier strength and degree of amplitude-modulation. The mean power in the carrier is the coefficient of $\cos \omega_0 t$ or $\sin \omega_0 t$ as t becomes infinite:

$$W_{f_0} = (A_0^2/2) \exp [-D_0^2\Omega(\infty)]\{1 + (kD_0/2) \lim_{f' \to 0} [\omega' Q(f')]\}, \tag{4.7}$$

showing that W_{f_0} is determined essentially by the type of angle-modulation. The second term of (4.7) vanishes for the usual types of phase-modulation, while for frequency-modulation the exponential factor vanishes, as explained earlier in section 2, reference 2, with the result that W_{f_0} is here described by the previous expression (2.21), reference 2. The continuum power is

$$W_c = (A_0^2/2)(1 + k^2\psi_A - \exp [-D_0^2\Omega(\infty)]). \tag{4.8}$$

Let us consider specifically the example of phase-modulation when the modulating noise waves are identical, so that

$$W_{AN}(f) = W_{\phi N}(f) = W_0 \exp (-\omega^2/\omega_b^2); \qquad \psi_A = \psi_F = W_0\omega_b/4\pi^{1/2} = \psi; \qquad \therefore \phi_A = \phi_F; \qquad r_0(t) = \exp [-\omega_b^2 t^2/4]. \tag{4.9}$$

To obtain the desired spectrum we need the following integrals:

$$\int_{-\infty}^{\infty} \begin{Bmatrix} \sin at \\ \cos at \end{Bmatrix} \exp (-b^2t^2 \pm ct) \, dt = (\pi^{1/2}/b) \exp [(c^2 - a^2)/4b^2] \begin{Bmatrix} \sin (\pm ac/2b^2) \\ \cos (\pm ac/2b^2) \end{Bmatrix}, \qquad Re\,(|\, b \,|) > 0. \tag{4.10}$$

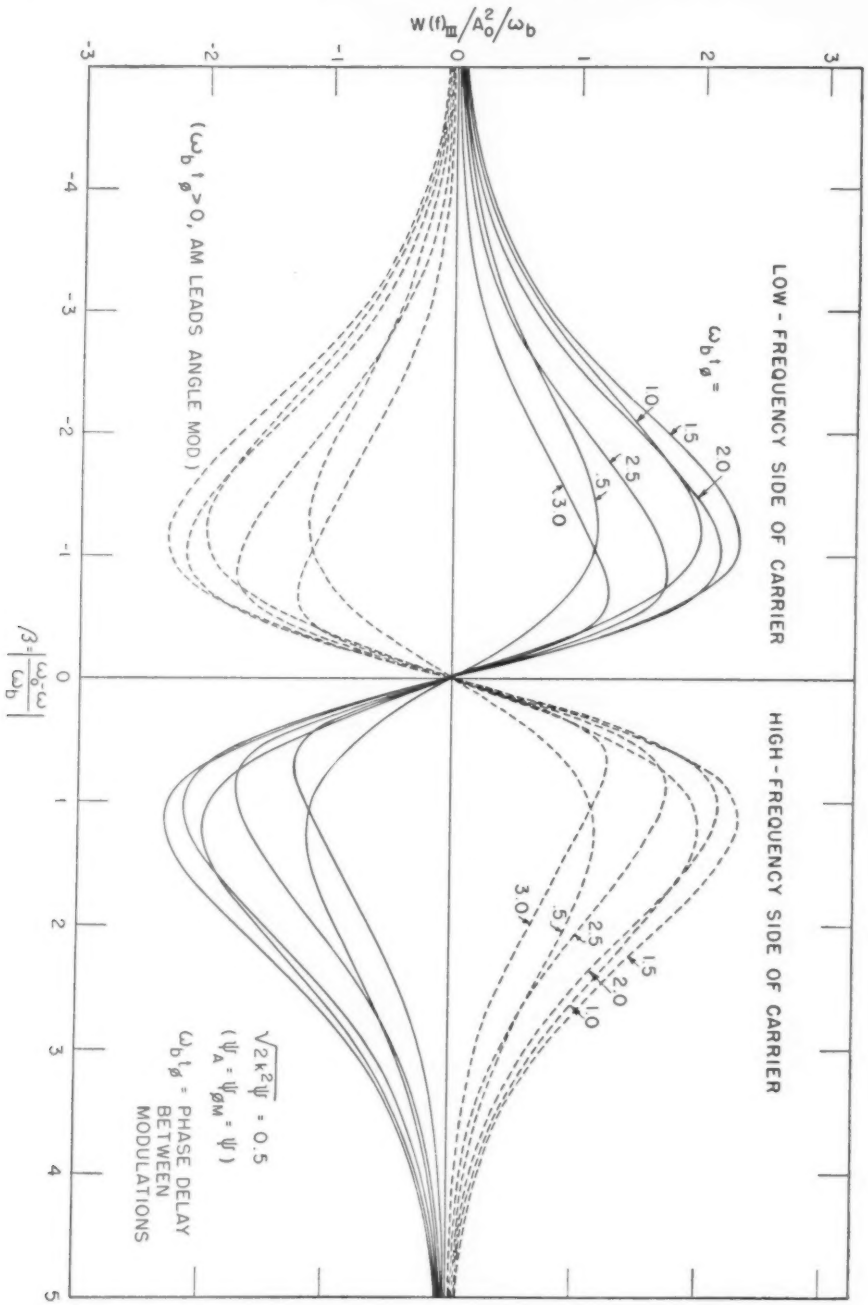


FIG. 4.2. Cross-spectral intensity of a carrier simultaneously amplitude and phase modulated by noise (no over-modulation).

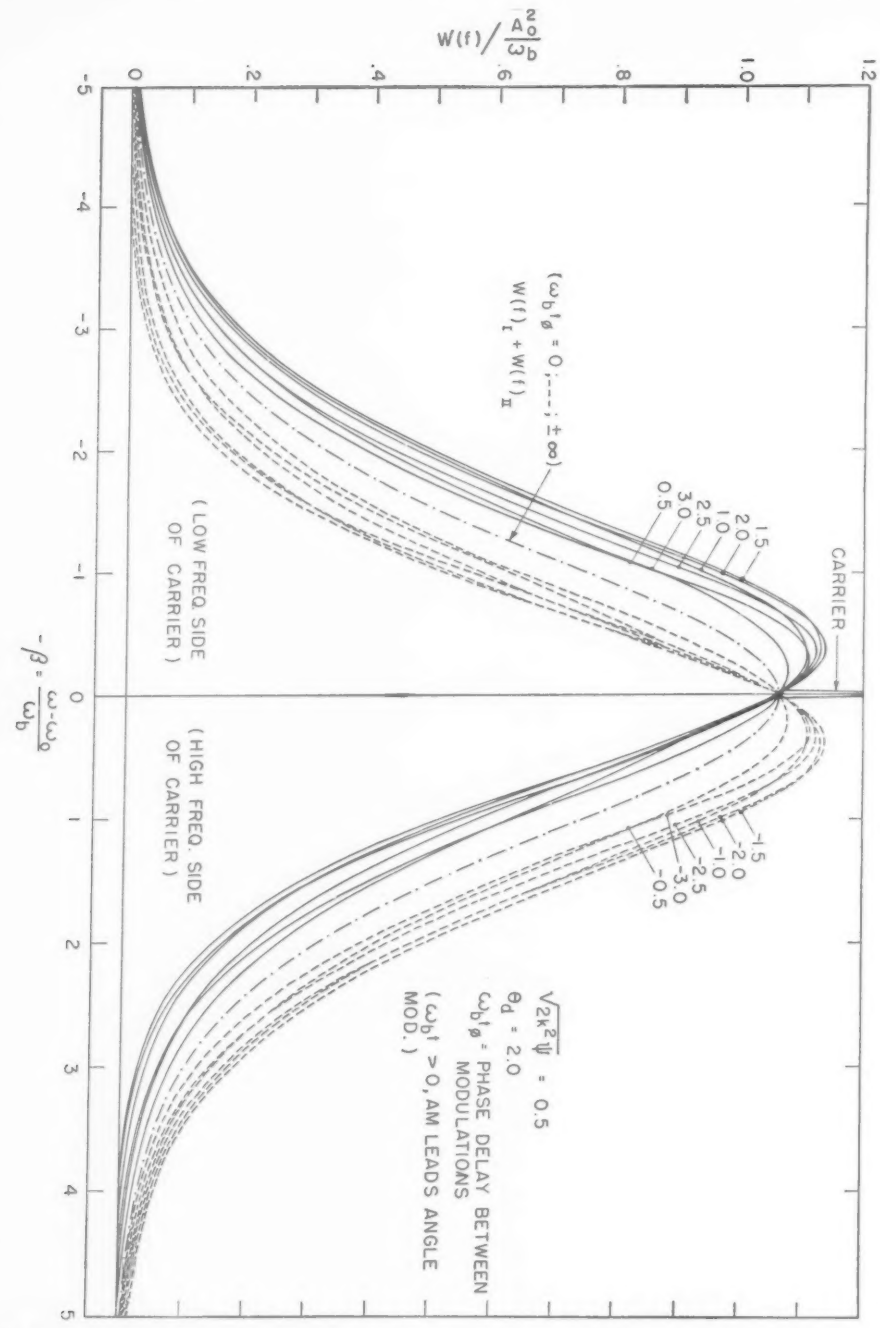


FIG. 4 3. Mean spectral intensity of a carrier simultaneously amplitude and phase modulated by correlated random noise (no over-modulation, θ_d constant, various delays).

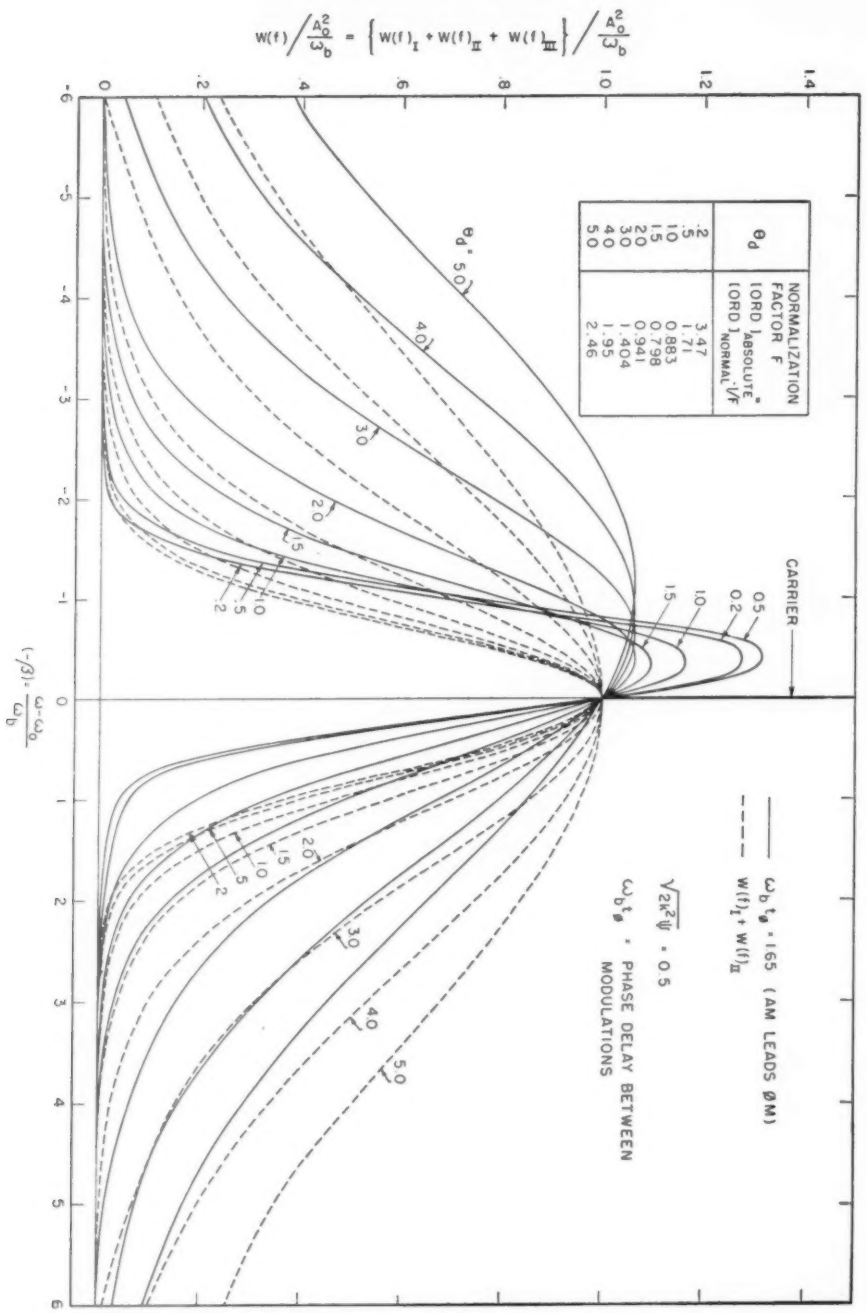


FIG. 4.4. Mean spectral intensity of a carrier simultaneously amplitude and phase modulated by correlated random noise (no over-modulation, t_ϕ constant, various degrees of modulation).

Application of the Wiener-Khintchine theorem yields

$$W(f) = W(f)_{\phi M} + W(f)_{\phi M \cdot AM} + W(f)_{\phi M \times AM} , \tag{4.11}$$

where $W(f)_{\phi M} = W(f)_{\mathrm{I}}$ is given by Eq. (2.48), reference 2, $W(f)_{\phi M \cdot AM} = W(f)_{\mathrm{II}}$ is obtained directly from the last term of (4.4) when $\omega_A = \omega_b$, and

$$W(f)_{\phi M \times AM} = W(f)_{\mathrm{III}} = \theta_d (k^2 \psi)^{1/2} \exp(-\theta_d^2)(A_0^2 \pi^{1/2}/\omega_b)$$
$$\cdot \sum_{n=0}^{\infty} \frac{\theta_d^{2n}}{n!(n+1)^{1/2}} \exp[-\beta^2/(n+1)] \left\{ 2 \sin\left(\frac{\omega_b t_\phi \beta}{n+1}\right) \exp[-n(\omega_b t_\phi)^2/4(n+1)] \right\}, \tag{4.12}$$
$$\beta = (\omega_0 - \omega)/\omega_b .$$

Typical spectra are shown in Figs. (4.2)-(4.4) for a variety of time-delays and for various values of θ_d . The spectra are in general asymmetrical, in the manner of Fig. (3.2); the spectral spread increases with the larger values of θ_d . We remark that, unlike the case of no coherence, it is not possible here to speak of a phase-modulating noise equivalent to a frequency-modulating wave (in the sense of Eq. (2.19), reference 2) which produces the same final spectrum, even if we assume the same disturbance causes both amplitude- and angle-modulations. The explanation lies in the entirely different types of modulation, which are coupled together in such a way that altering one does not produce a compensating change in the other. When the angle-modulation is sufficiently intense (i.e., high modulation indices), we obtain from (3.12) and (3.13) the needed asymptotic representation of the spectrum, namely

$$W(f) \simeq A_0^2 \sum_{q=0}^{\infty} \int_0^{\infty} dt \left\{ (1 + k^2 \psi_A r_0(t)_A) \cos(\omega_0 - \omega)t + 2kD_0 \sin(\omega_0 - \omega)t \right.$$
$$\left. \cdot \int_0^{\infty} Q(f', t_\phi) \sin \omega' t \, df' \right\} c^{(2q)}(t) \exp(-D_0^2 \Omega^{(2q)} t^2/2). \tag{4.13}$$

The precise forms of $\Omega^{(2)}$ etc. that appear in the exponent and in the polynomials $c^{(2)}(t)$, Eq. (3.13), are described in Eq. (2.34), reference 2. $W(f)_{\mathrm{I}}$ and $W(f)_{\mathrm{II}}$ follow from sec. 2, reference 2, and the above $W(f)_{\mathrm{III}}$ has the following general form

$$W(f)_{\mathrm{III}} \simeq -\pi k D_0 A_0^2 \sum_{m=0}^{\infty} \frac{Q_{2m+1}(t_\phi) \cdot G^{(2m+1)}(\beta)}{(2m+1)![D_0^2 \Omega^{(2)}]^{m+1}}, \tag{4.14}$$

where

$$Q_{2m+1}(t_\phi) \equiv \int_{-\infty}^{\infty} Q(f', t_\phi) \omega'^{2m+1} \, df',$$

and $G^{(2m+1)}(\beta)$ is the $(2m+1)$st derivative with respect to β of the expression in the parentheses of Eq. (2.33), reference 2. For phase- or frequency-modulation by noise with a gaussian spectrum one easily shows that when the amplitude- and angle-modulations are identical

$$W(f)_{\mathrm{III}} \Big|_{\phi M} \simeq (2\pi)^{3/2} (2k^2 \psi)^{1/2} \frac{A_0^2}{\omega_b \theta_d} \sum_{m=0}^{\infty} \frac{(-1)^m \phi^{(2m+1)}(\omega_b t_\phi / 2^{1/2})}{(2m+1)! \theta_d^{2m}} G^{(2m+1)}(\beta'') \tag{4.15}$$
$$\theta_d^2 \gg 1, \qquad \beta'' = \left(\frac{\omega_0 - \omega}{\omega_b}\right) \frac{2^{1/2}}{\theta_d}$$

or

$$W(f)_{III}\Big|_{FM} \backsimeq -\pi(2\pi)^{1/2}\frac{A_0^2}{\omega_d}[2(2k^2\psi)]^{1/2}\sum_{m=0}^{\infty}\frac{(-1)^m}{2^m}\left(\frac{\omega_b}{\omega_d}\right)^{2m}\frac{\phi^{(2m)}(\omega_b t_\phi/2^{1/2})}{(2m+1)!}$$
$$\cdot\, G^{(2m+1)}(\beta'); \qquad (\omega_b/\omega_d)^2 \ll 1, \qquad \beta' = \left(\frac{\omega_0-\omega}{\omega_b}\right)\left(\frac{\omega_b}{\omega_d}\right). \tag{4.16}$$

We observe that if the amplitude-modulation is particularly weak, the (angle $\times$ AM), or $W(f)_{III}$, components are strong compared to the incoherent terms $W(f)_{II}$, since the latter are $0(k^2\psi_A)$, whereas the former appear as $0(k^2\psi_A)^{1/2}$. The "pure" angle-modulation terms $W(f)_I$, however, remains unaffected.

5. Remarks on signal and noise modulations. When a signal accompanies the noise there are a number of interesting possibilities: (i) amplitude-modulation by signal and noise and angle-modulation by noise, (ii), amplitude-modulation by noise and angle-modulation by a signal and noise, and (iii), angle- and amplitude-modulations by signal and noise; in each case the noise waves and the signal waves are respectively coherent, but there is no coupling between signal and noise. We shall outline the principal steps in the process of obtaining the spectrum, but without giving a detailed treatment, inasmuch as the more complex problems of signal and noise in this case introduce no new concepts or techniques not already considered in the preceding sections.

Since (iii) contains (i) and (ii) as special cases, let us examine the more general case first. The modulated carrier is now

$$g(y) = V(t) = A_0(t)_{S+N}\exp\,[i(\omega_0 t + \Psi(t)_N + \Psi(t)_S)] \tag{5.1}$$

where $A_0(t)_{S+N}$ is an obvious generalization of (2.2a), viz.·

$$A_0(t)_{S+N} = (-A_0/2\pi)\int_C z^{-2}\,dz\,\exp\,\{iz[1 + kV_{AN}(t+t_\phi) + \mu V_S(t+t_\phi)]\} \tag{5.2}$$

and $\Psi(t)_N$, $\Psi(t)_S$ are specified by [(2.1), (2.2), (3.1b), ref. 2], or by (2.2). The autocorrelation function (2.3) becomes

$$R(t) = (A_0^2/2)\,Re\Big\{e^{-i\omega_0 t}(4\pi^2)^{-1}\int_C z^{-2}e^{iz}\,dz$$
$$\cdot\int_C \xi^{-2}e^{i\xi}\,d\xi\,[\exp\,\{ikV_{N1}z + ikV_{N2}\xi + i\Psi_{N1} - i\Psi_{N2}\}]_{\text{stat.av.}} \tag{5.3}$$
$$\cdot\,[\exp\,\{i\mu V_{S1}z + i\mu V_{S2}\xi + i\Psi_{S1} - i\Psi_{S2}\}]_{\text{stat.av.}}\Big\}$$

where the subscripts 1 and 2 refer to the wave at an initial and final time, t seconds apart. The first ensemble average has already been found, namely Eq. (2.6), and the second, which is the characteristic function for the joint signal modulations may be expressed as*

*It is assumed that the fundamental frequencies of the two signals are here commensurable; otherwise there will be no coherence between signals, and the characteristic function then factors into the product of two characteristic functions $F_2(z, \xi; t)_{S,AM}\cdot F_2(z, \xi, t)_{S,\text{angle }M}$, representing the (now) independent amplitude- and angle-modulations.

$$F_2(z, \xi; 1, -1; t)_S$$

$$= T_0^{-1} \int_0^{T_0} \exp\left[i\mu V_S(t_0 + t_\phi)z + i\mu V_S(t_0 + t_\phi + t)\xi + i\Psi(t_0)_S - i\Psi(t_0 + t)_S\right] dt_0 \quad (5.4)$$

$$= \sum_{m=0}^{\infty} \epsilon_m (-1)^m C_m(z) C_m(\xi) \cos m\omega_a t,$$

since F_2 is an even function of t. The auto-correlation function (2.18b) may still be used, provided that we replace the amplitude functions $h_{0,n+q}$ by $h_{m,n+q}$ according to (3.7) of ref. 1, with $C_m(z)$ in place of $B_m(z)$ therein, insert $(-1)^m \epsilon_m \cos m\omega_a t$ as an additional trigonometric factor, and perform the summation over all values of m. The precise form of $C_m(z)$ depends, of course, on the wave shape of the modulating signals, and except for the simple sinusoidal cases, are very complicated functions. In general, the spectrum will be asymmetrical in the discrete or signal portions of the distribution, as well as in the continuum.

Acknowledgment. The author wishes to express his sincere appreciation to Miss Virginia Jayne who performed the computations for this paper.

GRAVITY WAVES OF FINITE AMPLITUDE

III. Steady, Symmetrical, Periodic Waves in a Channel of Finite Depth*

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Introduction. Part I of this series** is concerned with the theory of finite amplitude gravity waves in a channel of infinite depth, Part II deals with the problem of the solitary wave. In the present paper, the finite amplitude wave problem is considered for a channel of finite depth and it will be shown that Parts I and II are special cases of this problem. The basic approach will not be explained in detail here since it is given in Part I, but the problem will be solved independently of the earlier parts.

1. Statement of problem. We deal with an incompressible fluid in a channel with a horizontal base, the fluid having been set in motion in such a way that the flow is two-dimensional and irrotational. The fluid motion is periodic with wave length λ in

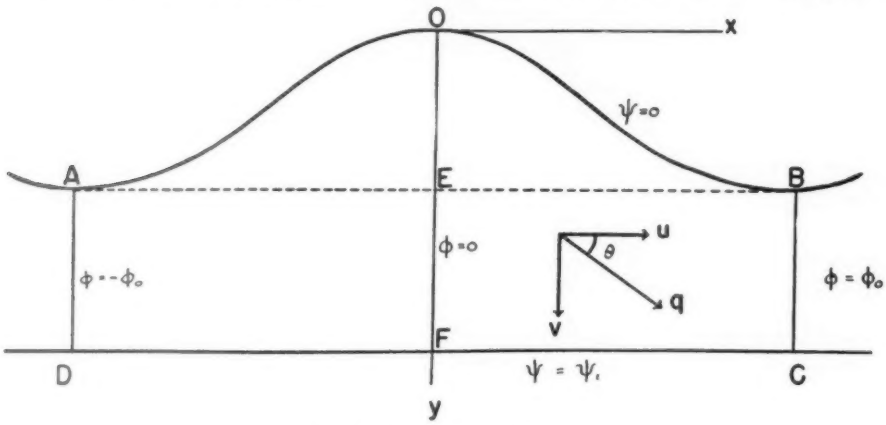

FIG. 1. $AB = \lambda$, $OE = a$, $AD = h$.

the horizontal x-direction, and is assumed to be symmetrical about a vertical line through any crest or trough of the wave pattern. It is assumed that the wave profile travels from right to left (see Figure 1) with velocity c. The axes Ox, Oy move with the crest O, Ox being horizontal and Oy vertically downwards. Relative to this set of axes the fluid motion is steady. A velocity potential ϕ and a stream function ψ exist for this motion and are defined by

$$d\phi = u\,dx + v\,dy,$$
$$d\psi = -v\,dx + u\,dy, \tag{1.1}$$

where (u, v) are the velocity components relative to $O(x, y)$.

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From the symmetry assumption it follows that ϕ is constant along the verticals through O, $B \cdots$. We take $\phi = 0$ along OF, $\phi = -\phi_0$ along AD and $\phi = \phi_0$ along BC. The lower boundary is a streamline which we take to be $\psi = \psi_1$ and the upper free surface is the streamline $\psi = 0$. Over the upper free surface the pressure p is everywhere constant. If q denotes the velocity of a fluid particle relative to $O(x, y)$ and θ its inclination to the x axis, then $u + iv = qe^{i\theta} = ce^{\tau + i\theta}$. Suppose q_0 is the relative velocity at O of the fluid particles, then Bernoulli's equation applied to the free surface leads to

$$q^2 = q_0^2 + 2gy, \qquad \psi = 0, \tag{1.2}$$

where q and y refer to values on the free surface. By differentiation with respect to ϕ Levi-Civita transforms this equation to the form

$$\frac{\partial \theta}{\partial \psi} = -\frac{g}{c^3} e^{-3\tau} \sin \theta, \qquad \psi = 0. \tag{1.3}$$

The method used here consists in replacing this condition by

$$\frac{\partial \theta}{\partial \psi} = -\frac{gl}{c^3} e^{-3\tau} \sin 3\theta, \qquad \psi = 0 \tag{1.4}$$

where l is a constant at our disposal. An alternative form of (1.4) is

$$\mathcal{I}\left\{ i \frac{d\xi}{dw} + \frac{gl}{c^3} e^{3\xi} \right\} = 0, \qquad \psi = 0 \tag{1.5}$$

where $\mathcal{I}$ denotes "imaginary part of", $w = \phi + i\psi$ and $\xi = -\tau + i\theta$.

The problem of the first approximation can now be stated as follows: ξ is a regular function of w in $0 \leq \psi \leq \psi_1$, is periodic and of period $2\phi_0$ in the ϕ direction, and is such that

$$\begin{aligned} &\text{(a)} && \mathcal{I}\{\xi\} = 0, && \psi = \psi_1 ; \\ &\text{(b)} && \mathcal{I}\{\xi\} = 0, && \phi = 0, \quad \phi = \pm\phi_0 ; \\ &\text{(c)} && \mathcal{I}\left\{ i \frac{d\xi}{dw} + \frac{gl}{c^3} e^{3\xi} \right\} = 0, && \psi = 0. \end{aligned} \tag{1.6}$$

In order to complete the statement of the problem it is necessary to postulate one further condition. In the infinitely deep channel problem and the solitary wave problem the actual fluid motion at infinity is specified ($\psi = \infty$ in the former, $\phi = \pm\infty$ in the latter) and in the present problem we must specify the actual flow at a convenient point of the fluid medium. This will be mentioned again later.

2. The method of solution. We first write $\chi = e^{-3\xi}$ and the problem in (1.6) may be restated as follows: χ is a regular function of w in $0 \leq \psi \leq \psi_1$, is periodic and of period $2\phi_0$ in the ϕ direction, and is such that

$$\begin{aligned} &\text{(a)} && \mathcal{I}\{\chi\} = 0, && \psi = \psi_1 ; \\ &\text{(b)} && \mathcal{I}\{\chi\} = 0, && \phi = 0, \quad \phi = \pm\phi_0 ; \\ &\text{(c)} && \mathcal{I}\left\{ \frac{i\, d\chi/dw - (3gl)/c^3}{\chi} \right\} = 0, && \psi = 0. \end{aligned} \tag{2.1}$$

Here, χ is defined within and on the boundaries of an infinite strip and we consider next how we may continue χ analytically throughout the whole of the w-plane. For this purpose we consider first of all the continuation of χ across $\psi = \psi_1$ into the strip $\psi_1 \leq \psi \leq 2\psi_1$. The line $\psi = 2\psi_1$ is the image of $\psi = 0$ in $\psi = \psi_1$; let P be any point of $\psi = 0$ and let Q be the image point of P in $\psi = \psi_1$. Suppose w_P is the value of w at P and w_Q that at Q. From Schwarz's principle of symmetry, if we wish to continue χ analytically across $\psi = \psi_1$, since $\vartheta(\chi) = 0$ on $\psi = \psi_1$ we must define the value of χ at Q, namely $\chi(w_Q)$, as follows

$$\chi(w_Q) = \overline{\chi(w_P)}. \tag{2.2}$$

Here the bar denotes the conjugate, that is, if $\chi(w_P) = \alpha + i\beta$ then $\chi(w_Q) = \alpha - i\beta$. The functions α and β vary with ϕ only as P moves along the line $\psi = 0$. The function χ is defined in a similar way throughout the upper strip. In terms of α and β, condition (2.1c) becomes

$$\alpha \frac{d\alpha}{d\phi} + \beta\left(\frac{d\beta}{d\phi} + \frac{3gl}{c^3}\right) = 0, \qquad \psi = 0. \tag{2.3}$$

It is easily verified that (2.3) is the condition also that the function

$$g(\chi) = \frac{[i\, d\chi/dw + (3gl)/c^3]}{\chi} \tag{2.4}$$

has zero imaginary part on $\psi = 2\psi_1$. Thus, if we write

$$f(\chi) = \frac{[i\, d\chi/dw - (3gl)/c^3]}{\chi} \tag{2.5}$$

then we have $\vartheta\{f(\chi)\} = 0$ on $\psi = 0$ and $\vartheta\{g(\chi)\} = 0$ on $\psi = 2\psi_1$, the only change being the reversal of the gravitational term. Thus we do not have perfect symmetry about $\psi = \psi_1$, but the result suggests, and we proceed tentatively from this point, that we build up the following image or periodic picture in the $\pm\psi$ directions (see Figure 2).

In the breaking case a zero of χ appears on $\psi = 0$ and we introduce, first of all, a zero of χ at $w = -i[(\pi/\nu) - 3\psi_1]$, where ν is a real parameter which is such that $\pi/\nu \geq 3\psi_1$. We introduce a similar zero at the reflection of this point in $\psi = \psi_1$, that is at $w = i[(\pi/\nu) - \psi_1]$. This we do because $\vartheta(\chi) = 0$ on $\psi = \psi_1$, in fact any distribution of singularities on $\phi = 0$ must be symmetrical about $\psi = \psi_1$. We do not consider the image of the zero at $w = -i[(\pi/\nu) - 3\psi_1]$ in the free surface $\psi = 0$ from the analogy with the solution in Part I, but consider next the "image" of the zero $w = i[(\pi/\nu) - \psi_1]$ in the free surface $\psi = 0$ where we must satisfy $\vartheta\{f(\chi)\} = 0$. The simple zero of χ at $w = i[(\pi/\nu) - \psi_1]$ will, in general, be a simple pole of $f(\chi)$ and we introduce, therefore, a simple pole at $w = -i[(\pi/\nu) - \psi_1]$. Similarly the "image" of the simple zero at $w = -i[(\pi/\nu) - 3\psi_1]$ in the line $\psi = 2\psi_1$ will be a simple pole at $w = i[(\pi/\nu) + \psi_1]$, and since this pole has to be reflected in $\psi = \psi_1$, it follows that the poles at $w = -i[(\pi/\nu) - \psi_1]$ and $w = i[(\pi/\nu) + \psi_1]$ are double poles. Proceeding further, the image of the pole at $w = -i[(\pi/\nu) - \psi_1]$ in $\psi = 2\psi_1$ is a zero at $w = i[(\pi/\nu) + 3\psi_1]$ while the pole at $w = i[(\pi/\nu) + \psi_1]$ will have as image a zero at $w = -i[(\pi/\nu) + \psi_1]$. We now introduce at $\psi = \psi_1 + 2\pi/\nu$ and $\psi = \psi_1 - 2\pi/\nu$ lines along which $\vartheta(\chi) = 0$ and at $\psi = 2\pi/\nu$ and $\psi = -2\pi/\nu + 2\psi_1$ lines along which $\vartheta\{f(\chi)\} = 0$ and $\vartheta\{g(\chi)\} = 0$ respectively. This

pattern is then repeated indefinitely in the $\pm\psi$ directions along $\phi = 0$. Similar patterns exist on all the lines $\phi = \pm 2m\phi_0$ where m is any integer. There are no zeros nor poles of χ at any other points in the w-plane.

Since we have made $\mathcal{I}\{f(\chi)\} = 0$ along $\psi = 2\pi/\nu$ and $\mathcal{I}\{g(\chi)\} = 0$ along $\psi = 2\psi_1$ it follows that $\mathcal{I}(\chi) = 0$ along $\psi = \psi_1 + \pi/\nu$, this being the converse of Schwarz's principle of symmetry. Hence $\mathcal{I}(\chi) = 0$ on the perimeter of the rectangle $FKHG$ and

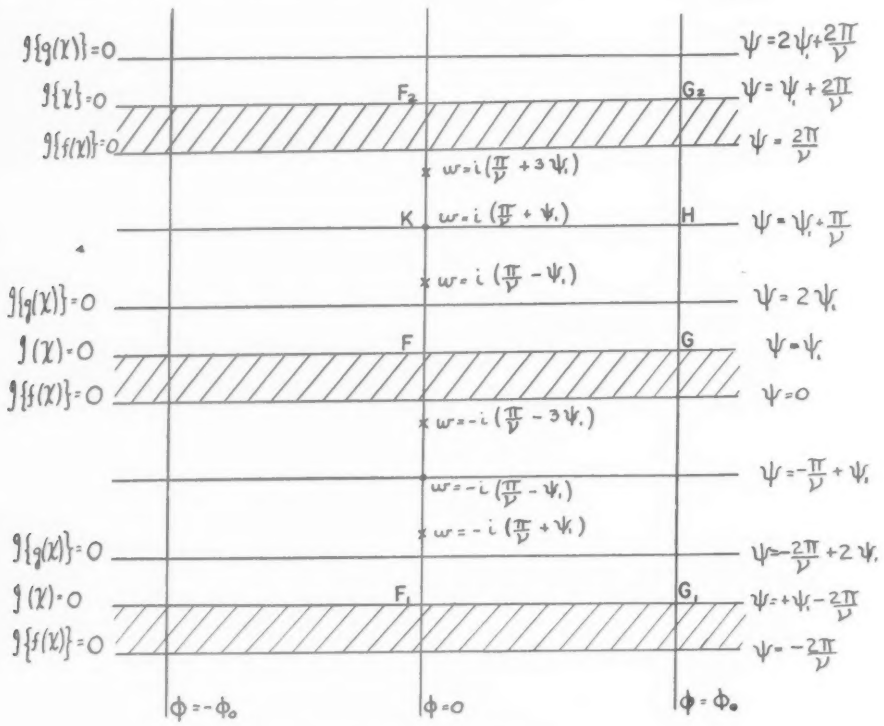

FIG. 2. x indicates a zero of χ, $\bullet$ indicates a pole of χ.

this rectangle can, therefore, be mapped on the upper half of the χ-plane and its perimeter into the real axis in the χ-plane. The mapping function is obtained from the Schwarz-Christoffel Theorem. Suppose the value of χ at F is A; at G let $\chi = A(1 - \mu)$ and at H let $\chi = A[1 - (\mu/k)^2]$, A, μ and k being real quantities. Then, since χ is infinite at K the transformation required is the solution of the differential equation

$$\frac{dw}{d\chi} = B\left(1 - \frac{\chi}{A}\right)^{-1/2}\left(\mu - 1 + \frac{\chi}{A}\right)^{-1/2}\left(\frac{\mu}{k^2} - 1 + \frac{\chi}{A}\right)^{-1/2}, \tag{2.6}$$

B being a complex constant in the general case. The appropriate solution of (2.6) is

$$\chi = A\left\{1 - \mu\,\mathrm{sn}^2\left[\frac{K}{\phi_0}(w - i\psi_1), k\right]\right\} \tag{2.7}$$

where sn (α, k) is* the usual Jacobian Elliptic Function with parameter k, whose real period is $4K$ and whose imaginary period is $2iK'$. In obtaining (2.7) from (2.6) the constant B has been so chosen as to make χ have period $2\phi_0$ in the ϕ direction; in this connection we use the relation sn $(\alpha + 2K) = -$ sn α. Furthermore the arbitrary constant of integration arising from the integration of (2.6) has been chosen so that χ is real when $\psi = \psi_1$; this enables us to satisfy (2.1a). We can determine μ as follows. Since the poles of sn (α, k) occur at points congruent to iK' and $2K + iK'$, on comparison with Figure 2, we obtain

$$\frac{\pi}{\nu} = \frac{\phi_0 K'}{K} \tag{2.8}$$

We know also from Figure 2 that a zero of χ occurs at $w = i[(\pi/\nu) - \psi_1]$ and hence

$$1 - \mu \operatorname{sn}^2\left[\frac{iK}{\phi_0}\left(\frac{\pi}{\nu} - 2\psi_1\right), k\right] = 0 \tag{2.9}$$

If we use (2.8) and the relation sn $(\alpha + iK') = 1/k$ sn α it follows from (2.9) that μ is given by

$$\mu = k^2 \operatorname{sn}^2\left(\frac{2iK\psi_1}{\phi_0}, k\right) \tag{2.10}$$

and thus the required solution is $\chi = e^{-3\xi} = e^{3r - 3i\theta}$ where

$$\chi = A\left\{1 - k^2 \operatorname{sn}^2\left(\frac{2iK\psi_1}{\phi_0}, k\right) \operatorname{sn}^2\left[\frac{K}{\phi_0}(w - i\psi_1), k\right]\right\}. \tag{2.11}$$

Before proceeding further we shall verify that χ in (2.11) satisfies all the conditions enunciated in (2.1). The conditions of regularity in $0 \leq \psi \leq \psi_1$, of periodicity in the ϕ direction and condition (2.1a) are certainly satisfied. To demonstrate that (2.1b) is satisfied we use Jacobi's Imaginary Transformation, namely

$$\begin{aligned} \operatorname{sn}(iu, k) &= \frac{i \operatorname{sn}(u, k')}{\operatorname{cn}(u, k')}, \\ \operatorname{cn}(iu, k) &= \frac{1}{\operatorname{cn}(u, k')}, \\ \operatorname{dn}(iu, k) &= \frac{\operatorname{dn}(u, k')}{\operatorname{cn}(u, k')}, \end{aligned} \tag{2.12}$$

where $k' = (1 - k^2)^{1/2}$ is the complementary modulus. It then follows, when $\phi = 0$, that χ is real; similarly when $\phi = \phi_0$ we have sn $[(K/\phi_0)(w - i\psi_1)] =$ sn $[K + (iK/\phi_0)(\psi - \psi_1)]$ and since sn $(\alpha + K) =$ cn α/dn α the results (2.12) apply and make χ real. Thus condition (2.1b) is satisfied on $\phi = 0$ and $\phi = \phi_0$ and elsewhere by periodicity. We consider finally condition (2.1c) or what is equivalent (1.4).

The addition formula for the sn function leads to the result

*When the parameter is not specified it will be understood that it is k.

$$\operatorname{sn}\frac{K}{\phi_0}(w - i\psi_1) = \frac{\operatorname{sn}(K\phi/\phi_0)\operatorname{cn}[iK(\psi-\psi_1)/\phi_0]\operatorname{dn}[iK(\psi-\psi_1)/\phi_0]}{1 - k^2\operatorname{sn}^2(K\phi/\phi_0)\operatorname{sn}^2[iK(\psi-\psi_1)/\phi_0]} + \frac{\operatorname{cn}(K\phi/\phi_0)\operatorname{dn}(K\phi/\phi_0)\operatorname{sn}[iK(\psi-\psi_1)/\phi_0]}{1 - k^2\operatorname{sn}^2(K\phi/\phi_0)\operatorname{sn}^2[iK(\psi-\psi_1)/\phi_0]} \tag{2.13}$$

From (2.12) we see that cn $(i\alpha)$ and dn $(i\alpha)$ are real while sn $(i\alpha)$ is imaginary, hence we obtain from (2.11)

$$\alpha = e^{3r}\cos 3\theta$$

$$= A\left\{1 - k^2\operatorname{sn}^2\left(\frac{2i\psi_1 K}{\phi_0}\right)\frac{\operatorname{sn}^2(K\phi/\phi_0)\operatorname{cn}^2[iK(\psi-\psi_1)/\phi_0]\operatorname{dn}^2[iK(\psi-\psi_1)/\phi_0]}{[1 - k^2\operatorname{sn}^2(K\phi/\phi_0)\operatorname{sn}^2[iK(\psi-\psi_1)/\phi_0]]^2} + \frac{\operatorname{cn}^2(K\phi/\phi_0)\operatorname{dn}^2(K\phi/\phi_0)\operatorname{sn}^2[iK(\psi-\psi_1)/\phi_0]}{[1 - k^2\operatorname{sn}^2(K\phi/\phi_0)\operatorname{sn}^2[iK(\psi-\psi_1)/\phi_0]]^2}\right\}, \tag{2.14}$$

$$\beta = ie^{3r}\sin 3\theta$$

$$= 2Ak^2\operatorname{sn}^2\left(\frac{2i\psi_1 K}{\phi_0}\right)\operatorname{sn}(K\phi/\phi_0)\operatorname{cn}(K\phi/\phi_0)\operatorname{dn}(K\phi/\phi_0)\operatorname{sn}[iK(\psi-\psi_1)/\phi_0]\times \frac{\operatorname{cn}[iK(\psi-\psi_1)/\phi_0]\operatorname{dn}[iK(\psi-\psi_1)/\phi_0]}{[1 - k^2\operatorname{sn}^2(K\phi/\phi_0)\operatorname{sn}^2[iK(\psi-\psi_1)/\phi_0]]^2}. \tag{2.15}$$

Using (2.14) and (2.15) we can evaluate $\partial\theta/\partial\psi$. Since $i\tan 3\theta = \beta/\alpha$ we have

$$3ie^{6r}\frac{\partial\theta}{\partial\psi} = \alpha\frac{\partial\beta}{\partial\psi} - \beta\frac{\partial\alpha}{\partial\psi} \tag{2.16}$$

and we substitute now for α and β from (2.14) and (2.15). To simplify the analysis we write

$$S = \operatorname{sn}\frac{K\phi}{\phi_0}, \qquad C = \operatorname{cn}\frac{K\phi}{\phi_0}, \qquad D = \operatorname{dn}\frac{K\phi}{\phi_0},$$
$$s_1 = \operatorname{sn}\frac{iK\psi_1}{\phi_0}, \qquad c_1 = \operatorname{cn}\frac{iK\psi_1}{\phi_0}, \qquad d_1 = \operatorname{dn}\frac{iK\psi_1}{\phi_0}. \tag{2.17}$$

We then obtain at $\psi = 0$,

$$\frac{3\phi_0 e^{6r}(\partial\theta/\partial\psi)(1 - k^2s_1^2S^2)^5}{2k^2KA^2\operatorname{sn}^2(2i\psi_1K/\phi_0)SCD}$$

$$= \left\{(1 - k^2S^2s_1^2)^2 - k^2\operatorname{sn}^2\left(\frac{2i\psi_1K}{\phi_0}\right)(S^2c_1^2d_1^2 + C^2D^2s_1^2)\right\}\{(c_1^2d_1^2 - s_1^2d_1^2 - k^2s_1^2c_1^2)\times$$
$$\times(1 - k^2s_1^2S^2) + 4k^2s_1^2c_1^2d_1^2S^2\} - 2k^2\operatorname{sn}^2\left(\frac{2i\psi_1K}{\phi_0}\right)s_1^2c_1^2d_1^2\times$$
$$\times\{(1 - k^2s_1^2S^2)[S^2(d_1^2 + k^2c_1^2) - C^2D^2] - 2k^2S^2(S^2c_1^2d_1^2 + C^2D^2s_1^2)\}. \tag{2.18}$$

The detailed reduction of the right hand side of (2.18) can be omitted here and we quote merely the final simplified version of (2.18), namely

$$\frac{3\phi_0 e^{6\tau}(\partial\theta/\partial\psi)(1-k^2S^2s_1^2)^2}{2k^2KA^2\,\mathrm{sn}^2[(2i\psi_1K)/\phi_0]SCD}=\frac{(1-2s_1^2+k^2s_1^4)(1-2k^2s_1^2+k^2s_1^4)}{1-k^2s_1^4}. \tag{2.19}$$

If we now use the formulae for sn $2u$, cn $2u$ and $dn\ 2u$ in terms of sn u, cn u and $dn\ u$ given in Whittaker and Watson's Modern Analysis p. 498 Ex. 5 it follows that the right hand side of (2.19) is simply

$$(1-k^2s_1^4)\ \mathrm{cn}\left(\frac{2i\psi_1K}{\phi_0}\right)\mathrm{dn}\left(\frac{2i\psi_1K}{\phi_0}\right)$$

so that

$$e^{6\tau}\frac{\partial\theta}{\partial\psi}\bigg|_{\psi=0}=\frac{2}{3}\frac{k^2KA^2}{\phi_0}\,\mathrm{sn}^2\left(\frac{2i\psi_1K}{\phi_0}\right)(1-k^2s_1^4)\ \mathrm{cn}\left(\frac{2i\psi_1K}{\phi_0}\right)\cdot\mathrm{dn}\left(\frac{2i\psi_1K}{\phi_0}\right)\frac{SCD}{(1-k^2s_1^2S^2)^2}. \tag{2.20}$$

From (2.15) we have

$$ie^{3\tau}\sin 3\theta\bigg|_{\psi=0}=-2Ak^2\,\mathrm{sn}^2\left(\frac{2i\psi_1K}{\phi_0}\right)\frac{SCDs_1c_1d_1}{(1-k^2s_1^2S^2)^2}, \tag{2.21}$$

hence combining (2.20) and (2.21) we obtain

$$\frac{\partial\theta}{\partial\psi}\bigg|_{\psi=0}=-\frac{1}{3}\left\{\frac{2iKA}{\phi_0}\frac{\mathrm{cn}\,[(2i\psi_1K)/\phi_0]\,\mathrm{dn}\,[(2i\psi_1K)/\phi_0]}{\mathrm{sn}\,[(2i\psi_1K)/\phi_0]}\right\}e^{-3\tau}\sin 3\theta. \tag{2.22}$$

If we now return to the fundamental boundary condition (1.11) we observe that it may be satisfied by (2.11) provided that

$$\frac{3gl}{c^3}=\frac{2iKA}{\phi_0}\frac{\mathrm{cn}\,[(2i\psi_1K)/\phi_0]\,\mathrm{dn}\,[(2i\psi_1K)/\phi_0]}{\mathrm{sn}\,[(2i\psi_1K)/\phi_0]}. \tag{2.23}$$

This is the formula for the velocity of propagation of the waves. Although the right hand side contains an i in the numerator it is real since cn (iu) and dn (iu) are real while sn (iu) is imaginary.

This solution of the problem is completed by defining A. If we postulate that the actual fluid velocity at $w=i\psi_1$ is U, the relative velocity at this point is $U+c$ hence from (2.11), since $\chi=e^{-3\xi}=(u-iv)^3/c^3$, we obtain

$$A=\left(\frac{U}{c}+1\right)^3. \tag{2.24}$$

The solution of the problem can now be written in the form

$$\left(\frac{dw}{dz}\right)^3=(U+c)^3\left\{1-k^2\,\mathrm{sn}^2\left(\frac{2i\psi_1K}{\phi_0},k\right)\mathrm{sn}^2\left[\frac{K}{\phi_0}(w-i\psi_1),k\right]\right\} \tag{2.25}$$

and the wave velocity formula in the form

$$\frac{3gl}{(U+c)^3}=\frac{2iK}{\phi_0}\frac{\mathrm{cn}\,[(2i\psi_1K)/\phi_0]\,\mathrm{dn}\,[(2i\psi_1K)/\phi_0]}{\mathrm{sn}\,[(2i\psi_1K)/\phi_0]}. \tag{2.26}$$

This wave velocity formula can be expressed in fairly simple terms by using the Jacobi and Landen Transformations. In the first place, applying the Jacobi transformation to the right hand side of (2.26) we obtain

$$\frac{3gl}{(U+c)^3} = \frac{2K}{\phi_0} \frac{\mathrm{dn}\,[(2\psi_1 K)/\phi_0,\, k']}{\mathrm{sn}\,[(2\psi_1 K)/\phi_0,\, k']\,\mathrm{cn}\,[(2\psi_1 K)/\phi_0,\, k']}\,. \tag{2.26'}$$

Landen's transformation states that

$$\mathrm{sn}\left\{(1+k')u, \frac{1-k'}{1+k'}\right\} = (1+k')\frac{\mathrm{sn}\,(u,k)\,\mathrm{cn}\,(u,k)}{\mathrm{dn}\,(u,k)}\,;$$

hence

$$\frac{3gl}{(U+c)^3} = \frac{2(1+k)K/\phi_0}{\mathrm{sn}\,\{2(1+k)K\psi_1/\phi_0\,,\,(1-k)/(1+k)\}}\,. \tag{2.27}$$

If we multiply both sides of this by ψ_1 the right hand side is then in non-dimensional form.

The breaking case is at once evident from Figure 2, for this corresponds to the case when the zero at $w = -i[(\pi/\nu) - 3\psi_1]$ lies on $\psi = 0$, that is the breaking case occurs when

$$\frac{\pi}{\nu} = 3\psi_1\,. \tag{2.28}$$

Using (2.8) this may be written in the form

$$\frac{K\psi_1}{\phi_0} = \frac{1}{3}K'. \tag{2.29}$$

This result is easily verified also by writing $\chi = 0$ in (2.11) and solving for $K\psi_1/\phi_0$ when $w = 0$. This quantity π/ν is one of the important parameters of the problem for we may note also that the small amplitude problem corresponds to large values of π/ν, when the system of singularities moves away from the region $0 \leq \psi \leq \psi_1$.

There are two points which must be mentioned before concluding this section, the first is that the problem of the uniqueness of the solution (2.11) has not been solved. Secondly, the problem of higher approximations has not been attempted, but it seems fairly certain that the higher approximations may be derived as in Part II by using the more exact boundary condition

$$\frac{\partial\theta}{\partial\psi} = -\frac{g}{c^3}\left\{\frac{1}{3}e^{-3\tau}\sin 3\theta + \frac{4}{81}e^{-3\tau}\sin^3 3\theta + \frac{16}{729}e^{-3\tau}\sin^5 3\theta + \cdots\right\} \tag{2.30}$$

and to assume a solution of the form

$$e^{-3\xi} = A\left\{1 + \mu_1\,\mathrm{sn}^2\left[\frac{K}{\phi_0}(w - i\psi_1)\right] + \mu_3\,\mathrm{sn}^6\left[\frac{K}{\phi_0}(w - i\psi_1)\right] + \cdots\right\}, \tag{2.31}$$

where μ_1 and μ_3 are constants. The results of Parts I and II suggest that $-\mu_1$ is the same as the μ of (2.10) and

$$\mu_3 = \frac{1}{54}\,\mathrm{sn}^6\left(\frac{2iK\psi_1}{\phi_0}\,,\,k\right);$$

also if μ_3 and higher terms are neglected, the results are likely to be in error by about 13% at the breaking point but will be much less for a non-breaking wave (here we assume that $l = 1/3$).

An alternative method of deriving the solution (2.11) is to make use of the isolated wave singularity distribution, given in Part II, and to use theta functions for the more extended distribution of the present periodic wave problem. This has been done, but is no shorter than the method expounded here.

3. Special cases of the solution (2.11). We demonstrate first that the problems of the infinitely deep channel and the solitary wave are merely particular cases of the solution (2.11) to the order of approximation in this paper. For this purpose we use the connections between the Jacobian Elliptic Functions and trigonometric functions, namely

$$\operatorname{sn}\left(\frac{2Kx}{\pi}, k\right) = \frac{2\pi}{Kk}\left\{\frac{q^{1/2}\sin x}{1-q} + \frac{q^{3/2}\sin 3x}{1-q^3} + \cdots\right\},$$

$$\operatorname{cn}\left(\frac{2Kx}{\pi}, k\right) = \frac{2\pi}{Kk}\left\{\frac{q^{1/2}\cos x}{1+q} + \frac{q^{3/2}\cos 3x}{1+q^3} + \cdots\right\},$$

$$\operatorname{dn}\left(\frac{2Kx}{\pi}, k\right) = \frac{\pi}{2K} + \frac{2\pi}{K}\left\{\frac{q\cos 2x}{1+q^2} + \frac{q^2\cos 4x}{1+q^4} + \cdots\right\},$$

where

$$\frac{1-k'^{1/2}}{1+k'^{1/2}} = \frac{2q + 2q^9 + 2q^{25} + \cdots}{1 + 2q^4 + 2q^{16} + \cdots}.$$

As k tends to zero, q tends to zero and we have in the limit

$$\lim_{k\to 0}\left(\frac{k}{q^{1/2}}\right) = 4. \tag{3.1}$$

Furthermore as $k \to 0$, $K \to \pi/2$ and $K' \to \infty$, hence

$$\lim_{k\to 0}\operatorname{sn}\left(\frac{2Kx}{\pi}, k\right) = \sin x, \quad \lim_{k\to 0}\operatorname{cn}\left(\frac{2Kx}{\pi}, k\right) = \cos x, \quad \lim_{k\to 0}\operatorname{dn}\left(\frac{2Kx}{\pi}, k\right) = 1. \tag{3.2}$$

Let us consider now what happens to the solution of the previous section as the parameter k tends to zero. When ϕ_0 is constant we see from (2.8) that π/ν tends to infinity, that is, the system of singularities moves to an infinite distance from the strip $0 \le \psi \le \psi_1$. In the problem of Part I, namely $\psi_1 = \infty$, there is a line of zeros parallel to $\psi = 0$ lying in the half plane $\psi \le 0$ and to reproduce the results of Part I we, therefore, make $k \to 0$ and $\psi_1 \to \infty$ in such a way that one line of zeros remains in the finite part of the w-plane. As $k \to 0$ we thus obtain

$$\operatorname{sn}\left\{\frac{K}{\phi_0}(w - i\psi_1)\right\} \to \frac{1}{2i}\{e^{i\pi w/2\phi_0}e^{\pi\psi_1/2\phi_0} - e^{-i\pi w/2\phi_0}e^{-\pi\psi_1/2\phi_0}\},$$

$$\operatorname{sn}\left(\frac{2i\psi_1 K}{\phi_0}\right) \to \frac{1}{2i}\{e^{-\pi\psi_1/\phi_0} - e^{\pi\psi_1/\phi_0}\};$$

hence as $k \to 0$ and $\psi_1 \to \infty$

$$\operatorname{sn}\left\{\frac{K}{\phi_0}(w - i\psi_1)\right\} \sim \frac{1}{2i} e^{i\pi w/2\phi_0} e^{\pi\psi_1/2\phi_0},$$

$$\operatorname{sn}\left(\frac{2i\psi_1 K}{\phi_0}\right) \sim -\frac{1}{2i} e^{\pi\psi_1/\phi_0}.$$

Thus from (2.11), as $k \to 0$, $\psi_1 \to \infty$

$$e^{-3\xi} \sim A\left\{1 - \left(\frac{1}{16} k^2 e^{3\pi\psi_1/\phi_0}\right) e^{i\pi w/\phi_0}\right\}. \tag{3.3}$$

The two limiting processes are independent of one another; if we now postulate that

$$\lim_{\psi_1 \to \infty} \lim_{k \to 0} \frac{1}{16} k^2 e^{3\pi\psi_1/\phi_0} = \text{constant} \tag{3.4}$$

and make $U = 0$ in (2.24), we obtain the solution of Part I, namely

$$e^{-3\xi} = 1 - Be^{i\pi w/\phi_0}. \tag{3.5}$$

The velocity of propagation formula (2.26) may be discussed in a similar way. With $U = 0$ the left hand side is $3gl/c^3$ and the right hand becomes

$$\frac{i\pi}{\phi_0} \frac{\cos\left[(\pi i\psi_1)/\phi_0\right]}{\sin\left[(\pi i\psi_1)/\phi_0\right]}$$

as $k \to 0$, so that as $\psi_1 \to \infty$ we obtain

$$\frac{3gl}{c^3} = \frac{\pi}{\phi_0} \tag{3.6}$$

which is similar to Part I.

Consider next what happens when we make k tend to its upper limit, namely unity. As $k \to 1$, $k' \to 0$, and thus we can deduce the relevant limits from Jacobi's Imaginary Transformation. We obtain from (2.8) and (2.12):

$$\operatorname{sn}\left\{\frac{K}{\phi_0}(w - i\psi_1), k\right\} = \operatorname{sn}\left\{\frac{\nu K'}{\pi}(w - i\psi_1), k\right\}$$

$$= i \frac{\operatorname{sn}\left\{(\nu K'/i\pi)(w - i\psi_1), k'\right\}}{\operatorname{cn}\left\{(\nu K'/i\pi)(w - i\psi_1), k'\right\}},$$

and as $k' \to 0$, since $K' \to (1/2)\pi$, we obtain

$$\lim_{k \to 1} \operatorname{sn}\left\{\frac{K}{\phi_0}(w - i\psi_1), k\right\} = \tanh\frac{1}{2}\nu(w - i\psi_1). \tag{3.7}$$

Similarly,

$$\lim_{k \to 1} \operatorname{sn}\left(\frac{2i\psi_1 K}{\phi_0}, k\right) = i \tan \nu\psi_1 . \tag{3.8}$$

It will be noted that by first substituting for the ratio K/ϕ_0 in the above expressions

we avoid the difficulty of indeterminacy of this ratio. The solution (2.11) now becomes

$$\begin{aligned} \chi &= A\{1 + \tan^2 \nu\psi_1 \tanh^2 \tfrac{1}{2}\nu(w - i\psi_1)\} \\ &= A \sec^2 \nu\psi_1\{1 - \sin^2 \nu\psi_1 \operatorname{sech}^2 \tfrac{1}{2}\nu(w - i\psi_1)\} \end{aligned} \tag{3.9}$$

which is the isolated wave solution of Part II, A being equal to $\cos^2 \nu\psi_1$. Similarly the wave velocity formula (2.26′) becomes

$$\frac{3gl}{c^3} = \frac{(2 \cos^2 \nu\psi_1)[(K'\nu)/\pi]}{\sin [(2\psi_1\nu K')/\pi] \cos [(2\psi_1\nu K')/\pi]} ;$$

hence

$$\frac{3gl}{c^3} = \nu \cot \nu\psi_1$$

as in Part II.

We can summarize the results of varying parameters as follows:

(a) $k \to 0$, $\psi_1 \to \infty$; finite amplitude waves in a channel of infinite depth;

(b) $k \to 1$; isolated wave;

(c) $k \to 0$; ψ_1 , ϕ_0 finite; small amplitude waves in a channel of finite depth;

(d) $K\psi_1/\phi_0 = \frac{1}{3}K'$; breaking condition at the crest.

(e) $0 \leq K\psi_1/K'\phi_0 \leq 1/3$.

The result (e) follows from expressing the condition that π/ν must lie between $3\psi_1$ and infinity. The result (c) follows from the early part of this section for in this case formula (2.11) becomes

$$e^{-3\xi} = A\left\{1 + k^2 \sinh^2 \frac{\pi\psi_1}{\phi_0} \sin^2 \frac{\pi}{2\phi_0}(w - i\psi_1)\right\}$$

and, k being small, this approximates to the usual solution for small amplitude waves. Furthermore the wave velocity formula in this case becomes

$$\frac{g}{c^3} = \frac{\pi}{\phi_0} \coth \frac{\pi\psi_1}{\phi_0}$$

and with the usual assumptions $\psi_1 = ch$, $\phi_0 = \frac{1}{2}c\lambda$ this becomes

$$c^2 = \frac{g\lambda}{2\pi} \tanh\left(\frac{2\pi h}{\lambda}\right).$$

—NOTES—

ON SOME STATISTICAL PROPERTIES OF HYDRODYNAMICAL AND MAGNETO-HYDRODYNAMICAL FIELDS*

By T. D. LEE (*University of California, Berkeley*)

1. Introduction. It is now recognized through the works of Taylor, Kolmogoroff, Heisenberg, Batchelor and many others that turbulence, rather than laminar flow, is statistically the most natural phenomenon exhibited by a non-conducting fluid. From the knowledge of magneto-hydrodynamical waves,[1] it is also believed that in a fluid of high electrical conductivity, magneto-turbulence may be the most natural phenomenon. The purpose of this paper is to study some of the statistical properties of a fluid and their relations to the theory of turbulence. The discussion of magneto-turbulence leads to the conclusion that in the usual Kolmogoroff region, where the behavior of the fluid is independent of detailed mechanisms of energy supply, the magnetic energy will be in equipartition with the kinetic energy and that the energy spectrum of either field will be proportional to $k^{-5/3}$ where k is the wave number.

The ergotic motion of an idealized fluid with no dissipation of heat is studied in section 2 and the existence of a Lionville Theorem is proved. The application of a dimensional consideration and some of these statistical properties to the steady state of magneto-turbulence is discussed in section 3.

2. Ergodic motion of an idealized fluid. Let us first consider an incompressible fluid with zero viscosity and zero electrical conductivity. If a certain amount of energy is given to this fluid, we wish to know statistically how this energy will be subdivided into various modes of possible motion. Let $\mathbf{v}(\mathbf{r})$, $p(\mathbf{r})$, ρ be the velocity, pressure and density of the fluid. The Euler's equation of motion gives†

$$\mathbf{v}^{\cdot} = -(\mathbf{v}, \nabla)\mathbf{v} - \frac{\nabla p}{\rho} \tag{1}$$

and

$$\nabla \cdot \mathbf{v} = 0. \tag{2}$$

It is convenient to analyze the motion of the fluid in terms of the Fourier components of its velocity. We write

$$p(\mathbf{r}) = \sum_{\mathbf{k}} p(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{r}} \tag{3}$$

$$\mathbf{v}(\mathbf{r}) = \sum_{\mathbf{k}} \mathbf{v}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{r}} = \sum_{\mathbf{k}} [\boldsymbol{\alpha}(\mathbf{k}) + i\boldsymbol{\beta}(\mathbf{k})] e^{i\mathbf{k}\cdot\mathbf{r}} \tag{4}$$

with $\quad p(\mathbf{k}) = p(-\mathbf{k})^*, \quad \mathbf{v}(\mathbf{k}) = \mathbf{v}(-\mathbf{k})^*, \quad \boldsymbol{\alpha}(\mathbf{k}) = \boldsymbol{\alpha}(-\mathbf{k})$

and $\quad \boldsymbol{\beta}(\mathbf{k}) = -\boldsymbol{\beta}(-\mathbf{k}).$

*Received April 2, 1951.

[1] H. Alfven, Arkiv. f. Mat. Astr. o. Fys. 29B, 2 (1943).

†A time derivative is indicated by a dot following the letter.

Equations (1) and (2) can then be written as

$$p(\mathbf{k}) = -\frac{\rho}{k^2}\sum_{\mathbf{k}'}[\mathbf{v}(\mathbf{k}+\mathbf{k}')\cdot\mathbf{k}][\mathbf{v}(-\mathbf{k}')\cdot\mathbf{k}], \tag{5}$$

$$\mathbf{v}^{\cdot}(\mathbf{k}) = -i\sum_{\mathbf{k}'}[\mathbf{v}(\mathbf{k}+\mathbf{k}')\cdot\mathbf{k}]\mathbf{v}(-\mathbf{k}') + \frac{i\mathbf{k}}{k^2}\sum_{\mathbf{k}'}[\mathbf{v}(\mathbf{k}+\mathbf{k}')\cdot\mathbf{k}][\mathbf{v}(-\mathbf{k}')\cdot\mathbf{k}], \tag{6}$$

and

$$\mathbf{v}(\mathbf{k})\cdot\mathbf{k} = 0. \tag{7}$$

For mathematical convenience we shall treat the three components of $\boldsymbol{\alpha}(\mathbf{k})$ and $\boldsymbol{\beta}(\mathbf{k})$ as independent but regard (7) as a constraint applied to the initial condition of the fluid. For from (6), if (7) is true at a particular moment it is always true at any other time.

Let us now consider a phase space with $\alpha_x(\mathbf{k})$, $\alpha_y(\mathbf{k})$, $\alpha_z(\mathbf{k})$, $\beta_x(\mathbf{k})$, $\beta_y(\mathbf{k})$, $\beta_z(\mathbf{k})$. $\cdots$, $\alpha_x(\mathbf{k}')$, $\alpha_y(\mathbf{k}')$, $\cdots$. as its coordinate axes.[2] In this space each point, compatible with the initial condition (7), represents a dynamical state of the fluid. The trajectory of this point governed by (6) describes the subsequent motion of the fluid. Differentiating (6), we have

$$\frac{\partial\alpha_i^{\cdot}(\mathbf{k})}{\partial\alpha_i(\mathbf{k})} + \frac{\partial\beta_i^{\cdot}(\mathbf{k})}{\partial\beta_i(\mathbf{k})} = 0; \qquad i = x, y, z. \tag{8}$$

This means that the density of a group of representative points in this phase space remain constant along their trajectories. Just as in an ordinary mechanical system, the existence of this Lionville Theorem suggests that the a priori probability of finding this system in any region of this phase space is proportional to the volume of this region.[3] From (6) the law of conservation of energy can be written as

$$\sum_{\mathbf{k}}[\boldsymbol{\alpha}^2(\mathbf{k}) + \boldsymbol{\beta}^2(\mathbf{k})] = \text{constant} \tag{9}$$

Thus, we find that statistically every mode of the Fourier components of velocity must be in energy equipartition. Let $F(k)$ be the spectrum of kinetic energy per unit volume. In the equilibrium case, we find

$$F(k) \propto k^2. \tag{10}$$

Next we discuss the corresponding case of an incompressible fluid with zero viscosity and infinite conductivity. Let $\mathbf{j}(\mathbf{r})$, $\mathbf{E}(\mathbf{r})$ and $\mathbf{H}(\mathbf{r})$ be the electric current, electric field and magnetic field respectively. The hydrodynamical equation becomes

$$\mathbf{v}^{\cdot} = -(\mathbf{v}, \nabla)\mathbf{v} - \frac{\nabla p}{\rho} + \frac{\mathbf{j}\times\mathbf{H}}{\rho c}. \tag{11}$$

Due to the infinite conductivity, we have

$$\mathbf{E} = -\frac{\mathbf{v}}{c}\times\mathbf{H}. \tag{12}$$

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[2] To avoid the difficulties of an infinite number of degrees of freedom, we may introduce in (3) and (4) an upper limit on k corresponding to, say, the inverse of atomic dimensions.

[3] Although the ergodic hypothesis may be true for a three dimensional fluid it is not true for a two-dimensional case. The existence of a conservation law on vorticity forbids the fulfillment of ergodic hypothesis for a two-dimensional fluid. See T. D. Lee, J. App. Phy. (in press).

Furthermore, $\mathbf{j}$, $\mathbf{E}$ and $\mathbf{H}$ are connected by the usual Maxwell Equations. On eliminating $\mathbf{j}$ and $\mathbf{E}$ from (11), (12) and the Maxwell Equations, the equation of motion of this fluid can be written as

$$\mathbf{v}^{\cdot} = -(\mathbf{v}, \nabla)\mathbf{v} - \frac{\nabla}{\rho}\left(p + \frac{\mathbf{H}^2}{8\pi}\right) + \frac{1}{4\pi\rho}(\mathbf{H}, \nabla)\mathbf{H}, \tag{13}$$

$$\mathbf{H}^{\cdot} = -(\mathbf{v}, \nabla)\mathbf{H} + (\mathbf{H}, \nabla)\mathbf{v} \tag{14}$$

and

$$\nabla\cdot\mathbf{v} = \nabla\cdot\mathbf{H} = 0. \tag{15}$$

The term

$$-\nabla\left(\frac{\mathbf{H}^2}{8\pi}\right) + \frac{1}{4\pi}(\mathbf{H}, \nabla)\mathbf{H}$$

in (13) is the force acting on the fluid due to Maxwell stress. In an infinitely conducting fluid the magnetic lines are always frozen in the matter. Equation (14) means that in a system moving with the matter the increment of $\mathbf{H}$ is given by the extension of the magnetic lines of forces due to relative motions of matter in different places.

As before, we make a Fourier expansion of the field variables. In addition to (3) and (4) we have

$$\mathbf{H} = \sum_{\mathbf{k}} \mathbf{H}(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{r}} = \sum_{\mathbf{k}} [\mathbf{g}(\mathbf{k}) + i\mathbf{h}(\mathbf{k})]e^{i\mathbf{k}\cdot\mathbf{r}} \tag{16}$$

with $\mathbf{g}(\mathbf{k}) = \mathbf{g}(-\mathbf{k})$ and $\mathbf{h}(\mathbf{k}) = -\mathbf{h}(-\mathbf{k})$.
In terms of these Fourier components (13), (14), (15) can be written as

$$\begin{aligned}\mathbf{v}^{\cdot}(\mathbf{k}) = i\sum_{\mathbf{k}'}\Big\{&-[\mathbf{v}(\mathbf{k}+\mathbf{k}')\cdot\mathbf{k}]\mathbf{v}(-\mathbf{k}') + \frac{1}{4\pi\rho}[\mathbf{H}(\mathbf{k}+\mathbf{k}')\cdot\mathbf{k}]\mathbf{H}(-\mathbf{k}')\Big\} \\ &+ \frac{i\mathbf{k}}{k^2}\sum_{\mathbf{k}'}\Big\{[\mathbf{v}(\mathbf{k}+\mathbf{k}')\cdot\mathbf{k}][\mathbf{v}(-\mathbf{k}')\cdot\mathbf{k}] - \frac{1}{4\pi\rho}[\mathbf{H}(\mathbf{k}+\mathbf{k}')\cdot\mathbf{k}][\mathbf{H}(-\mathbf{k}')\cdot\mathbf{k}]\Big\},\end{aligned} \tag{17}$$

$$\mathbf{H}^{\cdot}(\mathbf{k}) = i\sum_{\mathbf{k}'}\{-[\mathbf{v}(\mathbf{k}+\mathbf{k}')\cdot\mathbf{k}]\mathbf{H}(-\mathbf{k}') + [\mathbf{H}(\mathbf{k}+\mathbf{k}')\cdot\mathbf{k}]\mathbf{v}(-\mathbf{k}')\} \tag{18}$$

and

$$\mathbf{v}(\mathbf{k})\cdot\mathbf{k} = \mathbf{H}(\mathbf{k})\cdot\mathbf{k} = 0 \tag{19}$$

From (17)-(19) it can be verified readily that

$$\frac{\partial\alpha_i^{\cdot}(\mathbf{k})}{\partial\alpha_i(\mathbf{k})} + \frac{\partial\beta_i^{\cdot}(\mathbf{k})}{\partial\beta_i(\mathbf{k})} = 0,$$

and

$$\frac{\partial g_i^{\cdot}(\mathbf{k})}{\partial g_i(\mathbf{k})} + \frac{\partial h_i^{\cdot}(\mathbf{k})}{\partial h_i(\mathbf{k})} = 0; \qquad i = x, y, z. \tag{20}$$

The law of conservation of energy now becomes

$$\sum_{k}\left\{\frac{\rho}{2}[\alpha^2(\mathbf{k}) + \beta^2(\mathbf{k})] + \frac{1}{8\pi}[\mathbf{g}^2(\mathbf{k}) + \mathbf{h}^2(\mathbf{k})]\right\} = \text{constant}. \tag{21}$$

Let us now consider the phase space with $\alpha_x(\mathbf{k})$, $\beta_x(\mathbf{k})$, $g_x(\mathbf{k})$, $h_x(\mathbf{k})$, $\alpha_y(\mathbf{k})$, $\beta_y(\mathbf{k})$, $g_y(\mathbf{k})$, $h_y(\mathbf{k})$, $\cdots$, $\alpha_x(\mathbf{k}')$, $\cdots$. etc. as its coordinate axes. Every point in this space, compatible with (19), represents a dynamical state of this infinitely conducting fluid. Equation (20) suggests again that the a priori probability of finding this system in any region in this phase space is proportional to the volume of that region. Hence we find that in the equilibrium distribution every mode of the Fourier components of magnetic field and velocity field must be in energy equipartition. Let $M(k)$ be the corresponding energy spectrum of magnetic field per unit volume, then we have

$$M(k) = F(k) \propto k^2. \tag{22}$$

3. Turbulence and magneto-turbulence. In the case of a real fluid, due to the energy dissipation by either viscosity or electric resistance, a steady state is reached only if there is a constant source of energy supply. This situation can best be represented by a constant energy flow from small wave number regions to large wave number regions and then in turn a transformation into heat. The existence of this energy transfer makes it necessary that the energy spectrum must be quite different from the equilibrium case of an idealized fluid. However, since this energy transfer is only between different $|\mathbf{k}|$ values, it is expected that all modes of Fourier components of the same $|\mathbf{k}|$ values may be still in energy equipartition. We shall discuss only the statistical properties of magneto-turbulence in some detail as the corresponding properties of turbulence are well known and also can be obtained readily from the following discussion by putting $\mathbf{H} = 0$.

A. Steady case of an inviscid infinitely conducting fluid with a constant energy flow. Let us consider again an incompressible fluid with zero viscosity and infinite conductivity. If an amount of energy is supplied at a certain wave number, then due to the existence of non-linear terms in the equations (17) and (18), this energy is scattered between the magnetic field and the velocity field, and also is transferred within the different wave numbers of both fields. Due to the availability of more volume in phase space for high k values, statistically this energy is always transported toward higher and higher wave numbers. Let us now impose a boundary condition that a constant rate of energy ϵ per unit volume is supplied at wave numbers smaller than a certain k_1 and the same rate of energy is taken out at a k_2 much larger than k_1. Since the transfer of energy from low k to high k is of a statistical nature, it is expected that there exists a significant region, defined by limits ko and kc ($ko < kc$) between k_1 and k_2 such that in this region the behavior of the field depends only on ϵ but not on the boundary and the detailed mechanism of energy supply. This region will be called as "Kolmogoroff region" since Kolmogoroff[4] was the first one who pointed out the existence of such a region in ordinary turbulence. Let $E_v(k)$ and $E_H(k)$ be the rate of energy transfer per unit volume at k, within the velocity field and within the magnetic field respectively. In the Kolmogoroff region all functions can depend only on ϵ, ρ and k. From dimensional considerations the only forms that $F(k)$, $M(k)$, $E_v(k)$ and $E_H(k)$ can take are

$$F(k) = C_v \rho^{1/3} \epsilon^{2/3} k^{-5/3}, \tag{23}$$

$$M(k) = C_H \rho^{1/3} \epsilon^{2/3} k^{-5/3}, \tag{24}$$

$$E_v(k) = \kappa_v \epsilon, \tag{25}$$

[4] A. N. Kolmogoroff, A.N.C.R. Acad. Sci. U.R.S.S. **30**, 301 (1941); **32**, 16 (1941).

$$E_H(k) = \kappa_H \epsilon, \tag{26}$$

with

$$\kappa_v + \kappa_H = 1. \tag{27}$$

C_v , C_H , κ_v , κ_H are numerical constants. Equations (25) and (26) mean that the rate of energy transfer $E_v(k)$ and $E_H(k)$ must be constant at every k. Consequently, there can not be any energy transfer between the magnetic energy and kinetic energy in this Kolmogoroff region. From the result of previous section we have that for the equilibrium distribution every mode of the Fourier components of this idealized fluid must be in energy equipartition. A deviation from this distribution necessarily requires an energy flow. Hence, the presence of energy flow from low $|\mathbf{k}|$ to high $|\mathbf{k}|$ makes the spectra $F(k)$ and $M(k)$ different from k^2 law. Similarly, since there is no energy transfer between the magnetic field and the velocity field we can conclude that *for any fixed value of* $|\mathbf{k}|$ *the magnetic energy must be in equipartition with the kinetic energy.* We may write

$$C_v = C_H . \tag{28}$$

Similar arguments also lead to the conclusion that both magnetic field and velocity field are isotropic in the Kolmogoroff region. If the lower limit of Kolmogoroff region ko, is comparable to k_1 and the upper limit kc, is much greater than ko, one may normalize $F(k)$ in terms of the average velocity, v_0 , of the fluid. We have

$$\frac{\rho}{2} v_0^2 \cong \int_{k_0}^{\infty} F(k)\, dk.$$

Hence,

$$M(k) = F(k) \cong \frac{1}{3} \rho k_0^{2/3} v_0^2 k^{-5/3} \tag{29}$$

in the Kolmogoroff region.

B. Effects of viscosity and conductivity. The existence of viscosity and finite conductivity gives a particular mechanism of energy dissipation in the above discussion. Let μ and σ be the viscosity and electric conductivity respectively. We have the expression for energy dissipation per unit volume, $[\epsilon]_{\text{dissipation}}$.

$$[\epsilon]_{\text{dissipation}} = \mu \mid \nabla \times \mathbf{v} \mid^2 + \frac{\mathbf{j}^2}{\sigma} \cong \mu \mid \nabla \times \mathbf{v} \mid^2 + \frac{c^2 \mid \nabla \times \mathbf{H} \mid^2}{(4\pi)^2 \sigma} . \tag{30}$$

Equation (30) can be written in terms of the energy spectrum as

$$\left[\frac{dF(k)}{dt}\right]_{\text{viscosity}} = -\frac{2\mu k^2}{\rho} F(k) \tag{31}$$

and

$$\left[\frac{dM(k)}{dt}\right]_{\text{finite conductivity}} = -\frac{c^2 k^2}{2\pi\sigma} M(k). \tag{32}$$

The dependence on k^2 in (31) and (32) means that the effects of viscosity and finite conductivity are important only at high k values. The upper limit kc of the Kolmogoroff region may be estimated as the wave number at which the energy transfer between wave

numbers is comparable to the heat dissipation. From (17) and (18) the efficiency of energy transfer between different wave numbers can be estimated as

$$\frac{1}{\kappa_v}\left[\frac{1}{F(k)}\frac{dF(k)}{dt}\right]_{\text{transfer}} \simeq \frac{1}{\kappa_H}\left[\frac{1}{M(k)}\frac{dM(k)}{dt}\right]_{\text{transfer}} \simeq \sqrt{F(k)k^3/\rho} \tag{33}$$

Hence kc is given approximately by

$$\left(\frac{\mu}{\rho}+\frac{c^2}{4\pi\sigma}\right) \simeq \sqrt{F(k_c)k_c^3/\rho}. \tag{34}$$

Using (29), (34) can be written as

$$\frac{k_c}{k_0} \simeq R^{3/4} \tag{35}$$

where R is the modified Reynolds number defined by

$$R \equiv \rho v_0 \Big/ \left(\mu + \frac{c^2\rho}{4\pi\sigma}\right)k_0 \,. \tag{36}$$

Equation (35) means that the range of the Kolmogoroff region increases with increasing Reynolds number. The behavior of the energy spectrum between k_0, the lower limit of the Kolmogoroff region, and the boundary k_1 must depend on some extra-dimensional quantities related to the boundary or the detailed mechanism of energy supply. Hence no universal energy spectrum can be given in that region. The importance of this transition region will depend on the efficiency of energy coupling between the magnetic field and the velocity field and also on the detailed manner as how the energy is supplied. In ordinary turbulence it is assumed that such a transition is unimportant and this assumption is experimentally verified.[5] It may be natural to make the same assumption here. Due to the lack of experimental evidences, such assumption is always subject to criticisms and future verifications. But if this is the true case, then at high Reynolds number the total mechanical energy must be comparable to the total magnetic energy in a conducting fluid.

The author is indebted to Professors E. Fermi and S. Chandresekhar for discussions.

[5]See e.g. Heisenberg, Zeit. f. Phy. 124, 628 (1947).

ON THE INVERSION OF THE VOLTERRA INTEGRAL EQUATION*

By B. GROSS (*The National Institute of Technology, Rio de Janeiro*)

When the given kernel of the Volterra integral equation can be represented as a Laplace transform, the same representation is obtained for the resolving kernel of the equation. For this case the solution is given in explicit form.

The Volterra integral equation

$$f(x) = g(x) + \int_0^x k(x-y)g(y)\,dy \qquad (x > 0) \tag{1}$$

*Received May 8, 1951.

has proved to be of interest for the solution of many problems in physics. Its solution can be reduced to that of the simpler equation[1]

$$k(x) = m(x) + \int_0^x k(x - t)m(t)\, dt. \tag{2}$$

Here, $k(x)$ is the given kernel and $m(x)$ the resolving kernel of Eq. (1). Volterra[1] has given a solution of this equation in form of a series development. Doetsch[2] has solved it with the aid of the Laplace transform and Titchmarsh[3] with the aid of Fourier transforms. All these solutions however allow for very general conditions to be imposed upon the given kernel.

In physics integral equations of this type mainly occur in the theory of dielectric[4] and elastic[5] relaxation processes. In these applications $k(x)$ belongs to a very particular class of functions; it can be shown to satisfy conditions which will allow it to be represented as a Laplace transform

$$k(x) = \int_0^\infty r(s)e^{-xs}\, ds. \tag{3}$$

The purpose of the present paper is the discussion of the solution of Eq. (2) under the condition that $k(x)$ is given by the expression (3).

Equation (2) is of the convolution type. Application of the Laplace integral method of inversion[6] gives therefore

$$M(p) = \frac{K(p)}{1 + K(p)}, \tag{4}$$

where

$$K(p) = \int_0^\infty e^{-px}k(x)\, dx, \tag{5a}$$

$$M(p) = \int_0^\infty e^{-px}m(x)\, dx. \tag{5b}$$

It follows from (3) and (5) that

$$K(p) = \int_0^\infty e^{-px}\, dx \int_0^\infty r(s)e^{-xs}\, ds. \tag{6}$$

Suppose that $r(s)$ is a real function of integrable square over $(0, \infty)$. Then the order of integration in Eq. (6) can be inverted and

$$K(p) = \int_0^\infty \frac{r(s)}{s + p}\, ds. \tag{7}$$

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[1]V. Volterra and S. Pérès. *Théorie des fonctionelles*, Paris, Gauthier Villars, 1938.

[2]G. Doetsch, Math. Annalen **89**, 192 (1923).

[3]E. C. Titchmarsh, *Introduction to the theory of Fourier Integrals*, Oxford, Clarendon Press, 1937, p. 311.

[4]B. Gross, Zeits. f. Physik **107**, 217 (1937). Phys. Rev. **57**, 57 (1940).

[5]B. Gross, J. Applied Phys. **18**, 212 (1947); **19**, 257 (1948).

[6]R. V. Churchill, *Modern operational mathematics in engineering*. McGraw-Hill, New York, 1944, pg. 42.

$K(p)$ is therefore a Stieltjes transform. As such it is regular in the p plane cut along the negative real axis and the functions $K(pe^{\mp i\pi})$ are also of integrable square.

Applying the complex inversion theorem to Eq. (4) one obtains

$$m(z) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{K(p)}{1+K(p)}\, e^{pz}\, dp, \qquad (c > 0). \tag{8}$$

Suppose also that

$$|\, 1 + K(p)\,| > \delta > 0.$$

Then the line of integration can be deformed into a loop round the negative real axis, so that ultimately

$$m(z) = \frac{1}{2\pi i}\int_0^\infty \left[\frac{K(pe^{-i\pi})}{1+K(pe^{-i\pi})} - \frac{K(pe^{+i\pi})}{1+K(pe^{+i\pi})}\right] e^{-pz}\, dp. \tag{9}$$

Equation (9) can be written in the form

$$m(z) = \int_0^\infty \bar{r}(p) e^{-pz}\, dp. \tag{10}$$

This shows that under the given conditions the resolving kernel too is represented by a Laplace transform, and

$$\bar{r}(p) = \frac{1}{2\pi i}\left[\frac{K(pe^{-i\pi})}{1+K(pe^{-i\pi})} - \frac{K(pe^{+i\pi})}{1+K(pe^{+i\pi})}\right]. \tag{11}$$

Here, $K(pe^{\mp i\pi})$ is a shorthand way for writing $\lim_{\delta\to 0} K(-p \pm i\delta)$. Then, according to Eq. (7),

$$K(pe^{\mp i\pi}) = \lim_{\delta\to 0}\int_0^\infty \frac{r(s)}{s - p \pm i\delta}\, ds, \tag{12}$$

and

$$\lim_{\delta\to 0}\int_0^\infty \frac{r(s)}{s-p\pm i\delta}\, ds = \int_0^\infty \frac{r(s)}{s-p}\, ds \mp i\pi r(p), \tag{13}$$

where the integral denotes the principal value.[7] Substitution of (13) into (11) and decomposition into real and imaginary parts gives

$$\bar{r}(p) = \frac{r(p)}{\left[1 + \int_0^\infty \frac{r(s)}{s-p}\, ds\right]^2 + \pi^2 r^2(p)} \tag{14}$$

Equations (3), (10), and (14) represent the solution of the given equation.

The author is very indebted to Professor E. C. Titchmarsh whose help and advice has enabled him to write the present paper.

[7] The evaluation of this limit represents a particular case of the calculus of the boundary values of a function along a given path. The expression (13) is immediately obtained from the general formulae given e.g. by A. Hurwitz and R. Courant, *Funktionentheorie*, Julius Springer, Berlin 1929, p. 333.

ON CRACKS AND DISLOCATIONS IN SHAFTS UNDER TORSION*

By ALEXANDER WEINSTEIN (*University of Maryland*)

1. Introduction. Several decades ago W. Arndt [1] showed that the theory of shafts of revolution under torsion can be reduced to the investigation of axially symmetric motions of a fictitious incompressible fluid in a space of five dimensions. Let (x, y) be the meridian plane of the shaft, x being the axis of symmetry. All relevant quantities are determined by the potential $\Phi(x, y)$ and the stream function $\Psi(x, y)$ which satisfy the generalized Stokes-Beltrami equations

$$y^3\Phi_x = \Psi_y\,, \qquad y^3\Phi_y = -\Psi_x\,. \tag{1.1}$$

The displacement of a particle is perpendicular to the meridian plane and is given by the formula $v = y\Phi(x, y)$. The stresses in the meridian plane are $\tau_{\theta x} = y\Phi_x$ and $\tau_{\theta y} = y\Phi_y$, where θ denotes the polar angle in the plane perpendicular to the x-axis.** Any streamline $\Psi = \text{const.}$ is free from stress and can be taken as the profile of a free surface bounding the shaft.

By analogy with ordinary hydrodynamics, Arndt developed the method of sources and sinks in a space of five dimensions using however source distribution only on the axis of symmetry. Recently the method of sources and sinks has been extended by the present author to distributions outside of the axis of symmetry [2], [3]. This work was extended by A. Van Tuyl [4] and by M. A. Sadowsky and E. Sternberg [5]. The present paper gives new application of this theory to problems of cracks and dislocations in a shaft under torsion. A new and simple correspondence principle will be used connecting the method of sources and sinks with electrostatic problems in Generalized Axially Symmetric Potential Theory [6], [7] which will be denoted by the abbreviation GASPT.

2. The correspondence principle. GASPT deals with equations of the form

$$y^p\varphi_x(x, y) = \psi_y(x, y), \qquad y^p\varphi_y(x, y) = -\psi_x(x, y) \tag{2.1}$$

for all non-negative values of the parameter p. For $p = 1, 2, 3, \cdots$ φ represents an axially symmetric potential in the meridian plane of a space of $p + 2$ dimensions. The case $p = 0$ corresponds to harmonic functions in the plane. In all cases ψ is the corresponding stream function. For clarity's sake we shall sometimes write $\varphi\{p\}$ and $\psi\{p\}$ in place of φ and ψ. We have from (2.1) the following equations

$$\frac{\partial}{\partial x}\left(y^p\frac{\partial\varphi}{\partial x}\right) + \frac{\partial}{\partial y}\left(y^p\frac{\partial\varphi}{\partial y}\right) = 0, \qquad \frac{\partial}{\partial x}\left(y^{-p}\frac{\partial\psi}{\partial x}\right) + \frac{\partial}{\partial y}\left(y^{-p}\frac{\partial\psi}{\partial y}\right) = 0. \tag{2.2}$$

We see that φ and ψ satisfy the same equation only in the case $p = 0$, which corresponds to the well known fact that in plane hydrodynamics a stream function can be used as a velocity potential and vice versa. For $p \geqq 0$, we use a fundamental identity [6, p. 351, (48)], which can be written as follows

$$\psi\{p\} = Cy^{p+1}\varphi\{p + 2\}, \tag{2.3}$$

where C is an arbitrary constant. This identity permits us to obtain from a stream function $\psi\{p\}$ a certain potential $\varphi\{p + 2\}$ and vice versa.

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**We have put Lamé's constant μ equal to unity.

3. The method of sources and sinks. In the case of shafts, as well as in any other cases, solutions of certain problems are obtained by putting, for a constant U,

$$\Phi\{p\} = Ux - \varphi\{p\}, \qquad \Psi\{p\} = U(p+1)^{-1}y^{p+1} - \psi\{p\}. \tag{3.1}$$

In the usual cases φ and ψ are due to sources distributed in a finite domain. For the theory of shafts, p is to be taken equal to 3. Our main problem is to determine the lines $\Psi\{p\} = \text{constant}$, the line $\Psi\{p\} = 0$ being the line enclosing the singularities. We assume in the following that this line is closed. By (2.3), we may put $\psi\{p\} = U(p+1)^{-1}y^{p+1}\varphi\{p+2\}$. Using the notation $\varphi\{p+2\} = \varphi^*(x, y)$, we have (3.1)

$$\Psi\{p\} = U(p+1)^{-1}y^{p+1}(1 - \varphi^*(x, y)). \tag{3.2}$$

This shows that *to the closed streamline* $\Psi\{p\} = 0$ *correspond the level line* $\varphi^* = \varphi\{p+2\} = 1$. As $\psi\{p\}$ tends to $U(p+1)^{-1}y^{p+1}$ at infinity, we see from (3.2) that $\varphi\{p+2\}$ must tend to zero for $x^2 + y^2 \to \infty$. In this way the problem for $\Psi\{p\}$ is now reduced by our correspondence principle to the electrostatic problem for $\varphi\{p+2\}$.

4. The crack problem. A flat interior circular crack perpendicular to the axis of symmetry of the shaft is defined as a circular disc separating the material. It is assumed that no stress is acting across the surface of the disc. The definition of the crack adopted here is the same as that used in another problem by M. A. Sadowsky and E. Sternberg [8]. Let us note that A. E. Green [9] uses the word crack in a different sense, namely in order to denote the displacement on the surface of a body due to localized pressure.

Let $x = 0, y = 0$ be the center of our disc of radius b. By Arndt's theory, the boundary condition is given by $\Psi\{3\} = 0$ on both sides of the segment $x = 0, 0 \leqq y \leqq b$ in the upper half of the meridian plane, Ψ being an even function of x. The problem is, by (3.2), reduced to the determination of the electrostatic potential $\varphi\{5\}$ of a disc of radius b in a space of seven dimensions. This problem will be now solved by using ellipsoidal coordinates. Let

$$\frac{x^2}{\lambda} + \frac{y^2}{\lambda + b^2} = 1 \tag{4.1}$$

be a family of confocal ellipses which generate through a rotation about the x-axis a family of oblate spheroids $\lambda = \text{constant}$ in a space of seven dimensions. The value $\lambda = 0$ corresponds to the disc. Let us put

$$\varphi^* = \frac{8b^5}{3\pi}\int_\lambda^\infty \frac{du}{(u + b^2)^3(u)^{1/2}} \tag{4.2}$$

Then φ^* satisfies (2.2) for $p = 5$, vanishes at infinity and takes the constant value one on the disc. The integral in (4.2) can be expressed in terms of elementary functions, the result being the formula

$$\varphi^* = \frac{2}{\pi}\operatorname{arccot}\frac{(\lambda)^{1/2}}{b} - \frac{2b}{3\pi}\frac{(3\lambda + 5b^2)(\lambda)^{1/2}}{(\lambda + b^2)^2}. \tag{4.3}$$

By Sec. 3 we have

$$\psi\{3\} = 4^{-1}Uy^4\varphi^* \tag{4.4}$$

In order to obtain $\varphi\{3\}$ we introduce in the x, y-plane elliptic coordinates by the formulae

$$x = b \sinh \xi \cos \eta, \qquad y = b \cosh \xi \sin \eta. \tag{4.5}$$

By comparison with (4.1), see that

$$b^2 \sinh^2 \xi = \lambda, \qquad b^2 \cosh^2 \xi = \lambda + b^2 \tag{4.6}$$

As (4.5) defines a conformal transformation, i.e. a local rotation and uniform dilation of the coordinate system, we have, in place of (2.1)

$$y^p \varphi_\xi = \psi_\eta, \qquad y^p \varphi_\eta = -\psi_\xi, \tag{4.6}$$

where by (4.5), $y = b \cosh \xi \sin \eta$.
By (4.3) and (4.4) we have

$$\psi\{3\} = Ub^4(6\pi)^{-1}\{3 \cosh^4 \xi \operatorname{arccot}(\sinh \xi) - 3 \sinh^3 \xi - 5 \sinh \xi\} \sin^4 \eta. \tag{4.7}$$

To find φ, we use (4.6) for $p = 3$, which yields φ_η. By an integration we obtain finally

$$\varphi\{3\} = Ub\pi^{-1}\left\{\frac{2}{3 \cosh^2 \xi} + 2 \sinh \xi \operatorname{arccot}(\sinh \xi) - 2\right\} \cos \eta. \tag{4.8}$$

Obviously φ vanishes for $x + y^2 \to \infty$.

The formulae (4.8) and (4.7) together with (3.1), taken for $p = 3$ give the solution of the crack problem in elliptic coordinates. The same solution can be also expressed in terms of x and y. In fact, the quantity

$$s(x, y) = [x^2 + |y - b|^2]^{1/2} + [x^2 + |y + b|^2]^{1/2} \tag{4.9}$$

is a function $f(\lambda)$ of λ alone. Putting $y = 0$ in (4.1) we find $\lambda = x^2$. Therefore on the ellipse $\lambda =$ constant we have $s = s(x, 0) = 2(x^2 + b^2)^{1/2} = 2(\lambda + b^2)^{1/2}$. This equation yields

$$\lambda = \tfrac{1}{4}s^2(x, y) - b^2. \tag{4.10}$$

This value of λ can be substituted in (4.6), (4.7), and (4.8) and gives us explicit formulas for $\varphi(x, y)$ and $\psi(x, y)$ in terms of x and y.

The same functions φ and ψ can also be expressed in terms of integrals involving products of Bessel functions. However, such integrals are less suitable for the discussion. In fact, these new expressions should be rather considered as new formulae yielding certain integrals in terms of elementary functions. The corresponding results (also for an arbitrary value of the parameter p) will be published elsewhere.

5. Discussion of the results in the crack problem. For the discussion, we have to use the formulae for displacements and stress given in the Introduction, and the formulas for Φ, Ψ, φ and ψ given in paragraphs 3 and 4. It will suffice to note the following results. The transverse displacement $v(x, y)$ is given for $x = 0$ by the formulae

$$v(0, y) = \begin{cases} \pm\dfrac{4}{3\pi} Uy(b^2 - y^2)^{1/2}, & \text{for} \quad 0 \leqq y \leqq b, \\ 0, & \text{for} \quad y > b. \end{cases} \tag{5.1}$$

The signs $+$ and $-$ correspond here and in the following to the positive and to the negative side of the y-axis respectively. The stress $\tau_{\theta x}$ is zero on the disc, i.e. for $x = 0$, $0 \leqq y \leqq b$ but becomes infinite like $(y - b)^{-1/2}$ for $y > b$. The stress $\tau_{\theta y}$ tends to infinity in the same manner for $x = 0$, $y \nearrow b$. For large values of $x^2 + y^2$, φ^* tends to zero like $(x^2 + y^2)^{-5/2}$. Therefore the formula $\Psi = 4^{-1}Uy^4(1 - \varphi^*)$ shows that already for

moderate values of y, a streamline Ψ = constant is essentially parallel to the axis of symmetry, so that we have a nearly cylindrical beam with a circular crack.

It is clear that the method presented here can be immediately applied to cavities of the form of oblate spheroids.

6. Dislocations. According to the classical theory of Volterra a surface of dislocation is a surface of discontinuity for the displacements. The stresses remain continuous across this surface and the displacements on both sides of a dislocation differ only by a rigid body motion. In the case of a shaft under torsion we take as the surface of the dislocation the same circular disc as in the crack problem. The only admissable dislocation is, in this case, a rigid rotation of the negative side of the disc relative to its positive side. Such a dislocation is readily obtained by taking in (3.1) for $\varphi = \varphi\{3\}$ the potential of a uniform magnetic shell in a five dimensional space. By a specialization of a general formula [7, p. 76, (6.1)] we have

$$\varphi = Mb^2y^{-1}\int_0^\infty e^{-xt}J_1(yt)J_2(bt)\,dt, \qquad x \geqq 0 \tag{6.1}$$

and

$$\psi = -Mb^2y^2\int_0^\infty e^{-xt}J_2(yt)J_2(bt)\,dt, \qquad x \geqq 0. \tag{6.2}$$

For $x \leqq 0$, we have $\varphi(-x, y) = -\varphi(x, y)$, $\psi(x, y) = \psi(-x, y)$. In (6.1) and (6.2) M is a positive constant and the J_ϱ denote Bessel functions. In view of the results given in [7] we have the following values for the displacement v in the plane of the disc:

$$v(0, y) = \begin{cases} \pm My, & \text{for} \quad y < b \\ 0, & \text{for} \quad y > b \end{cases} \tag{6.3}$$

Moreover we have

$$\lim v(x, y) = Mb(\pi - \beta)\pi^{-1}$$

for $x \to 0$, $y \to b$ and $(b - y)x^{-1} \to \cot\beta$, where $x \geqq 0$ and $0 \leqq \beta \leqq \pi$. By a well known property of uniform magnetic shells which remain valid in any number of dimensions, the stresses $\tau_{\theta z}$ and $\tau_{\theta y}$ remain continuous across the disc. However, they become infinite in the neighborhood of its rim like $(y - b)[x^2 + (b - y)^2]^{-1}$ and $x[x^2 + (b - y)^2]^{-1}$ respectively. The connection of the formulae (6.1) and (6.2) with elliptic integrals analogous to those given in [4] and [5] will be given elsewhere. Further applications of "generalized electrostatics" are given in [10].

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INDUCED MASS WITH FREE BOUNDARIES*

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The concept of induced mass, and some of its properties, are extended to the case of an incompressible liquid having a free surface. The usual[1] treatment of the case of a non-viscous fluid extending to infinity must be considerably changed to do this.

1. Minimum principle. Let R be a region filled with an incompressible liquid, bounded in part by a *wetted wall* W, and in part by a *free surface* S at constant pressure p_0. The region R moves with the liquid, and may extend to infinity in some directions.

Suppose the fluid *accelerated from rest*, by an acceleration of W whose inward normal component toward the liquid is an arbitrary function $f(\mathbf{x})$ of position. Letting $\mathbf{u} = \mathbf{u}(\mathbf{x}; t)$ denote liquid velocity and $\mathbf{a} = \partial\mathbf{u}/\partial t$ denote acceleration, clearly

$$\text{Div } \mathbf{a} = \Sigma\partial(\partial u_k/\partial t)/\partial x_k = \partial(\Sigma\partial u_k/\partial x_k)/\partial t = 0, \tag{1}$$

by incompressibility. Similarly, the Navier-Stokes equations with *gravity neglected* are[2]

$$Du_i/Dt = -\partial p/\rho\partial x_i + \nu\nabla^2 u_i\,. \tag{2}$$

Since $\mathbf{u} = 0$ initially, the *initial* acceleration therefore satisfies $a_i = Du_i/Dt = -\partial p/\rho\partial x_i$, or, setting $A = (p_0 - p)/\rho$,

$$\mathbf{a} = \nabla A, \qquad \text{where} \qquad p = p_0 - \rho A, \qquad \textit{initially}. \tag{3}$$

Combining (3) with (1), we get

$$\nabla^2 A = 0 \qquad \text{in } R. \tag{4}$$

The free surface condition is simply

$$A = 0 \qquad \text{on} \qquad S. \tag{5}$$

Finally, by continuity, we have

$$a_{\text{normal}} = \partial A/\partial n = f(\mathbf{x}) \qquad \text{on } W. \tag{6}$$

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[1] Given in [1], Chap. V; in [3]; and in Chap. VI of Lamb's *Hydrodynamics*.

[2] Here and below D/Dt is the substantial derivative $\partial/\partial t + u_k\,\partial/\partial x_k$, while $\partial/\partial n$ denotes the inward normal derivative on the surface of the liquid.

By potential theory,[3] conditions (4) to (6) uniquely determine A; we shall call A the *acceleration potential* associated with the given acceleration of W; by (3), A determines p.

THEOREM 1. The *acceleration kinetic energy*

$$T = \frac{1}{2}\rho \iiint_R \nabla A \cdot \nabla A \, dR \tag{7}$$

is minimized by the free surface condition (5), relative to all other volume-conserving flows in R satisfying (6).

Remark. The acceleration kinetic energy is half the second time derivative of the ordinary kinetic energy.

Proof. Let $\nabla A + \mathbf{b}$ be any other volume-conserving flow satisfying (6). Then

$$\text{Div } \mathbf{b} = \text{Div } \mathbf{b} + \nabla^2 A = \text{Div } (\nabla A + \mathbf{b}) = 0$$

Consider now the expanded acceleration kinetic energy

$$T = \frac{1}{2}\rho \int_R (\nabla A + \mathbf{b}) \cdot (\nabla A + \mathbf{b}) \, dR$$

$$= T_0 + \rho \int_R (\nabla A \cdot \mathbf{b}) \, dR + \frac{1}{2}\rho \int_R (\mathbf{b} \cdot \mathbf{b}) \, dR.$$

Since the last term is positive unless $\mathbf{b}$ vanishes identically, the theorem will be proved if we can show that the middle integral is zero. But since Div $\mathbf{b} = 0$, clearly

$$\text{Div } (A \cdot \mathbf{b}) = A \text{ Div } \mathbf{b} + (\nabla A) \cdot \mathbf{b} = \nabla A \cdot \mathbf{b}.$$

Hence, by the Divergence Theorem, letting b_n denote the outward normal component of $\mathbf{b}$,

$$\int_R (\nabla A \cdot \mathbf{b}) \, dR = -\int_W A b_n \, dS - \int_S A b_n \, dS. \tag{8}$$

Since $\nabla A + \mathbf{b}$ satisfies (6) on W, $b_n = 0$ on W and the first surface integral in (8) vanishes. By (5), the second surface integral vanishes; hence the proof is complete.

When S is void, our result reduces to the classical case ([2], p. 84). There is however no relation between the kinetic energy for *steady* flow with a free boundary, which is always infinite in an infinite non-viscous stream, and virtual mass. In this respect, the case of flows with free boundaries is unlike the classical case.

2. General interpretation. The acceleration potential A described in Sec. 1 has a very general interpretation. Let a fluid motion, defined by an "undisturbed" velocity field $\mathbf{U} = \mathbf{U}(x; t)$, be altered at time $t = 0$ by an additional instantaneous normal acceleration $f(\mathbf{x})$ of W. Because of incompressibility, we have for the perturbed velocity field $\mathbf{U} = \mathbf{u}$,

$$\text{Div } \mathbf{u} = 0, \qquad \text{whence} \qquad \text{Div } \mathbf{a} = 0, \tag{9}$$

where $\mathbf{a} = \partial \mathbf{u}/\partial t$ denotes the additional acceleration.

[3] For existence and uniqueness theorems, see O. D. Kellogg, *Potential theory*, pp. 218, 315. Although such theorems have been proved only for a restricted class of infinite regions, there is no reason to doubt their general validity.

Subtraction of the Navier-Stokes equations for $\mathbf{U}$ from those for $\mathbf{U} + \mathbf{u}$ gives

$$\frac{\partial u_i}{\partial t} = -u_k \frac{\partial U_i}{\partial x_k} - U_k \frac{\partial u_i}{\partial x_k} - u_k \frac{\partial u_i}{\partial x_k} - \frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \nabla^2 u_i \,. \tag{10}$$

Here p denotes the increment in the pressure field due to the additional acceleration of W; moreover, this is true even if gravity is considered. Since the perturbation begins at $t = 0$, clearly $\mathbf{u}(x; 0) = 0$, whence (10) reduces to

$$a_i = -\partial p/\rho \partial x_i \,, \qquad \text{at} \qquad t = 0. \tag{11}$$

Since the rest of the reasoning leading to (4) to (7) applies, we get the following result.

THEOREM 2. The instantaneous pressure distribution required to accelerate a moving incompressible fluid is the same as if the fluid were at rest.

The preceding result can be extended to the case of an "impulsive" velocity change, by integrating (10) over a short interval of time, which is then allowed to tend to zero. It is readily seen that if $\mathbf{u}$ is uniformly bounded and $\iint \nabla^2 \mathbf{u}\, dR\, dt = o(1)$ under these circumstances, and if the normal impulse per unit area is defined as $p^* = \mathrm{Lim}_{\Delta t \to 0} \int_0^{\Delta t} p dt$, we get the limiting analog of equation (3),

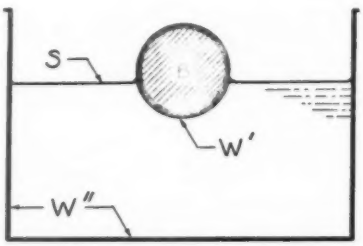
$$u_i = -\partial p^*/\rho \partial x_i \qquad \text{on} \qquad W. \tag{3*}$$

From (9), the fact that $p^* = 0$ on the free surface S, and continuity we get similarly equations analogous to (4) to (6). From these we deduce the following result.

THEOREM 3. The impulse $p^*(\mathbf{x})$ per unit area required to produce an additional normal "impulsive velocity" $u_n = f(\mathbf{x})$ of W, is $-\rho A$, where A is the acceleration potential of Sec. 1.

In the case of the irrotational motion of a non-viscous fluid, since $\nabla^2 \mathbf{u} = 0$, the assumption about $\nabla^2 \mathbf{u}$ (which is hard to prove rigorously) is superfluous.

3. Induced mass tensor. To define an *induced mass tensor*, let $W = W' + W''$ consist in part of the wetted area W' of an accelerated *missile*, and otherwise of the fixed walls W'' of a *container*. The case of a ball, floating on the surface of the water in a pail (Fig. 1) is typical.



We let A^1, A^2, A^3 denote the acceleration potentials in R for unit translations of W' parallel to the axes; let A^4, A^5, A^6 denote those for unit rigid rotations of W' about the coordinate axes; in all cases we assume $\partial A/\partial n = 0$ on W'', and $A = 0$ on S. We then define the symmetric 6×6 induced mass tensor (matrix) $\| T_{hk} \|$, as in the ordinary case ([1], p. 154), by

$$T_{hk} = \rho \iiint_R \nabla A^h \cdot \nabla A^k \, dR = T_{kh} \tag{12}$$

Our aim is to show that the T_{hk} have most of their familiar properties. Although the preceding definition of the T_{hk} was given in principle by L. I. Sedov [4], he did not deduce the properties proved below.

First, note that the diagonal components T_{hh} of induced mass, satisfy the conditions of Theorem 1. From this fact the following result follows immediately.

COROLLARY 1. The diagonal components T_{hh} of induced mass are increased if either (i) the region R occupied by liquid is increased (i.e., the free boundaries are pushed out), or (ii) free surface area is replaced by container walls or wetted missile area.

COROLLARY 2. Let a volume ΔV of liquid be replaced by missile. The new translation induced mass T'_{hh} satisfies the inequality.

$$T'_{hh} \geq T_{hh} - \rho \Delta V \tag{13}$$

Proof. Let $R' = R - \Delta V$ denote the reduced volume occupied by liquid. Consider the acceleration field of R, under which ΔV is accelerated as a rigid body under h-translation, while R' is given the acceleration corresponding to T'_{hh} . By Theorem 1, $T'_{hh} + \rho \Delta V$ will exceed T_{hh} , proving formula (13). This result can be extended to rotational components, if $\rho \Delta V$ is replaced by the appropriate rigid moment of inertia.

Again, we have by Green's second identity

$$-T_{hk} = \rho \int_S A^h \frac{\partial A^k}{\partial n}\, dS + \rho \int_{W'} A^h \frac{\partial A^k}{\partial n}\, dS + \rho \int_{W''} A^h \frac{\partial A^k}{\partial n}\, dS$$

By (5), $A^h = 0$ on S, and so the first summand is zero; by (6), $\partial A^k/\partial n = 0$ on W'', hence the last summand is zero.

Using (3) and reversing signs, we therefore have

$$T_{hk} = \int_{W'} (p_h - p_0) \frac{\partial A^k}{\partial n}\, dS, \tag{14}$$

where p_0 is the free surface pressure, and p_h is the pressure on W' under unit h-acceleration from rest. Also on W', $\partial A^k/\partial n = \partial x^k/\partial n$ $[k = 1, 2, 3]$ and $\partial A^4/\partial n = x_2 \partial x_3/\partial x_n - x_3 \partial x_2/\partial x_n$ etc., by definition. Hence, cancelling out the effect of the constant pressure p_0 over the missile, we have

THEOREM 4. The tensor component T_{hk} represents the total h-component of initial *pressure force* required to produce a unit k-acceleration of the missile.

In view of the linearity of (3) to (6) in A and p, we deduce immediately the following corollary.

COROLLARY 1. If, at any instant, the configuration and flow field are given, the instantaneous effect of an additional acceleration of the missile, with components $a_1, \cdots a_6$, is to produce components $\sum_{k=1}^{6} T_{hk} a_k$ of reaction in the liquid.

This result will be compared with experimental data in Sec. 5.

4. Momentum interpretation. If there is no container, so that $W = W'$ and $W'' = 0$, momentum interpretations of the T_{hk} are also possible. I shall show this directly from momentum considerations. (To make this rigorous, one must consider, as in [1] or [3], the convergence of the momentum integrals involved.)

Let the missile, supposed of finite diameter, be given initial acceleration from rest, and let C by any cylinder parallel to the x_h-axis, whose finite cross-section contains the wetted area $W = W'$ of the missile. The total h-momentum[4] in C is finite; moreover,

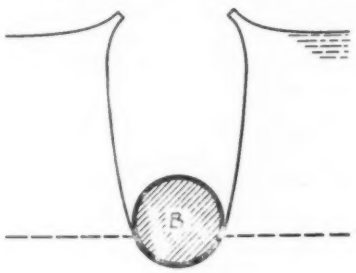
[4]By h-momentum, we mean linear momentum parallel to the x_h-axis.

since the liquid is at rest initially, the rate of convection of fluid h-momentum out of C is zero. Since the fluid is non-viscous, the shear stresses across this boundary are also zero; hence the rate of total transfer of h-momentum across the boundary of C is zero. We conclude that the time rate of increase of h-momentum of the liquid in C, is equal to the h-component of pressure force.

A corresponding result holds for moments of momentum; in this case C must be a solid of revolution containing W, with the axis about which the moment is taken, for axis of symmetry. Combining with Theorem 2, we get the following result. (We define an "h-curve" to be a straight line parallel to the x_h-axis, if $h = 1, 2, 3$; and to be a circle perpendicular to the x_{h-3}-axis, with center on the x_{h-3}-axis, if $h = 4, 5, 6$.)

THEOREM 5. Let a missile be given a unit k-acceleration from rest, in a liquid bounded by the missile wetted area and a free surface. Then the rate of increase of the h-component of liquid momentum in any region C bounded by h-curves, which contains W, is exactly T_{hk} initially.

5. Application. we can apply the corollary of Theorem 4 to the case of a missile B travelling vertically in a cavity, as in Fig. 2. The reaction to acceleration of B will be at least as great as if the liquid were confined to the underside of a plane bounding the wetted area of B (indicated as a dashed line in Fig. 2).



In this case, as first noted by von Kármán [6], the acceleration potential can be obtained by symmetry, using the reflection principle. It follows that the instantaneous inertia opposed by the liquid to the acceleration of B, should be *between* 50 per cent *and* 100 per cent *of that offered if there were no cavity, and the same area were wetted.*

This is not necessarily the same as the change in the cavity drag coefficient due to *steady* acceleration or deceleration (cf. Sec. 1, end). In fact, data obtained at the Naval Ordnance Laboratory[5], indicate that this change probably corresponds to less than 25 per cent of that occurring if there were no cavity.

The preceding results also apply to the case of impact on water, for which the momentum interpretation of §4 may be of interest. In this case, a rigorous lower bound to the loss of energy at impact is obtained by Theorem 1. Unfortunately, about 50 per cent of the energy lost at impact is presumably absorbed by the energy of compression (and radiated as acoustic energy); hence this bound is excessively low.

However, as the impact phase has been extensively discussed by other authors ([4], [5], [6]), especially in connection with the landing of seaplane floats, we shall not discuss it further here.

[5] A. May and J. Woodhull, "The virtual mass of a sphere entering water vertically", *Journal of Applied Physics* 21 1285-9 (1950).

6. Correction for hydrostatic force. Since gravity has been neglected above, it is interesting to have a rough estimate of the effect of gravity on the pressure exerted by a liquid on a missile moving through it with wetted area W' trailed by a cavity (Fig. 2). We suppose the liquid incompressible, and bounded by W', container walls W'', and a free surface S. The additional instantaneous acceleration $\mathbf{b}$ due to a vertical gravity field with intensity g satisfies $\mathbf{b} = g\nabla B$, where $\nabla^2 B = 0$, $B = y$ (depth coordinate) on S, and $\partial B/\partial n = 0$ on $W' + W'' = W$; the associated hydrostatic pressure is $\rho g(y - B)$.

For given boundary configurations S and W, the resulting "hydrostatic acceleration potential" gB can be most easily estimated using an electrolytic tank, and the results interpreted in terms of the dimensionless *cavity buoyancy coefficient*

$$C_H = \frac{\text{hydrostatic bouyancy force}}{\rho g \times \text{mean depth} \times \text{horizontal projection of } W'}$$

In this way, C_H was estimated[7] for three two-dimensional cavity flows, having profiles similar to that of Fig. 2. The *cavity buoyancy coefficients averaged about* 25 per cent.

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ON THE NON-UNIQUENESS OF PERIODIC SOLUTIONS FOR AN ASYMMETRIC LIÉNARD EQUATION*

By G. F. D. Duff and N. Levinson (*Massachusetts Institute of Technology*)

The following result has been stated by H. Serbin [5, Theorem II]. Let $f(x)$, $g(x)$ be continuous for $-\infty < x < \infty$, and let

$$f(x) < 0, \qquad -x_1' < x < x_1, \tag{1.0}$$

$$f(x) > 0, \qquad x < -x_1', \qquad x_1 < x, \tag{1.1}$$

where $x_1' > 0$ and $x_1 > 0$. Let

$$\int_0^\infty f(x)\,dx > 0 \tag{1.2}$$

$$xg(x) > 0, \qquad x \neq 0 \tag{1.3}$$

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and let either $\int_0^\infty f(x)\,dx$ or $\int_0^\infty g(x)\,dx$ diverge. Then the equation

$$\ddot{x} + f(x)\dot{x} + g(x) = 0, \qquad \dot{x} = dx/dt, \tag{1.4}$$

has a unique periodic solution.

In case $g(x) \equiv x$ and $f(x)$ is even (1.4) has been treated by Liénard [3]. In case $f(x)$ is even and $g(x)$ is an odd function (1.4) has been treated by Levinson and Smith [2]. Here an example will be given with $g(x) \equiv x$ and with $f(x)$ meeting the requirements (1.0), (1.1) and (1.2) where at least three periodic solutions for (1.4) exist thus showing that the result of Serbin is incorrect as stated.

We note that if $f(x)$ satisfies the required conditions so does $\epsilon f(x)$ for any $\epsilon > 0$. The equation to be considered here is

$$\ddot{x} + \epsilon f(x)\dot{x} + x = 0 \tag{1.5}$$

where $f(x)$ is a polynomial. If $x = r\cos\theta$ and $\dot{x} = -r\sin\theta$ then (1.5) yields $\dot{r} = \epsilon r f(r\cos\theta)\sin^2\theta$, $\dot{\theta} = 1 + \epsilon f(r\cos\theta)\sin\theta\cos\theta$ from which follows

$$\frac{dr}{d\theta} = \frac{\epsilon r f(r\cos\theta)\sin^2\theta}{1 + \epsilon f(r\cos\theta)\sin\theta\cos\theta}. \tag{1.6}$$

For each solution of (1.6) $r = r(\theta)$ periodic of period 2π there corresponds a periodic solution of (1.5).

Since with $\epsilon = 0$ the solutions of (1.6) $r = \text{const.}$, are bounded and since the right side of (1.6) is analytic in r, θ and ϵ it follows from the successive approximations existence theorem, as indicated for example in Lefschetz [1, Chap. II] that the solutions of (1.6) with $r = \rho$ for $\theta = 0$ are given by $r = H(\theta, \rho, \epsilon)$ where H is analytic in (θ, ρ, ϵ) for any finite range such as $0 \leq \theta \leq 2\pi$, $\frac{1}{2} \leq \rho \leq 5$, and $|\epsilon|$ sufficiently small. Thus H can be expanded in powers of ϵ yielding

$$r = H_0(\theta, \rho) + \epsilon H_1(\theta, \rho) + \epsilon^2\gamma(\theta, \rho, \epsilon)$$

where $|\gamma|$ is uniformly bounded for the above range of θ and ρ for $|\epsilon|$ small. Using $r = H$ in (1.6) it follows, equating powers of ϵ on each side of the equation, that $H_0(\theta, \rho) = \rho$ and

$$H_1(\theta, \rho) = \int_0^\theta \rho f(\rho\cos\sigma)\sin^2\sigma\,d\sigma.$$

If

$$F(\rho) = \int_0^{2\pi} \rho f(\rho\cos\sigma)\sin^2\sigma\,d\sigma \tag{1.7}$$

then

$$H(2\pi, \rho, \epsilon) - \rho = \epsilon F(\rho) + \epsilon^2\gamma(2\pi, \rho, \epsilon). \tag{1.8}$$

If $H(2\pi, \rho, \epsilon) - \rho$ vanishes for some ρ then as already stated (1.5) has a periodic solution passing through the point $x = \rho$, $\dot{x} = 0$ in the phase plane of the variables $(x, \dot{x})$. Let $\rho_0 > 0$ and let $F(\rho_0) = 0$ and let $F(\rho)$ change sign at $\rho = \rho_0$. Then for a given small $\delta > 0$, it follows from (1.8) that $H(2\pi, \rho, \epsilon) - \rho$ has opposite signs for $\rho = \rho_0 - \delta$ and $\rho = \rho_0 + \delta$ providing ϵ is small enough. Since $H(2\pi, \rho, \epsilon) - \rho$ is continuous in ρ this implies that $H(2\pi, \rho, \epsilon) - \rho$ vanishes for some ρ between $\rho_0 - \delta$ and $\rho_0 + \delta$.

It follows therefore that if it is shown that $F(\rho)$ in (1.7) has 3 positive zeros and

changes sign at each zero then for small ϵ (1.5) has at least three periodic solutions as stated. [4]

Let

$$f(x) = A_3x^6 - A_2x^4 + A_1x^2 - A_0 - Cx \tag{1.9}$$

where A_i and C are constants. Let

$$I_k = \int_0^{2\pi} \sin^2\sigma \cos^{2k} \sigma \, d\sigma, \qquad k = 0, 1, 2, 3.$$

Then $I_k > 0$ and from (1.7) it follows that

$$F(\rho) = A_3I_3\rho^7 - A_2I_2\rho^5 + A_1I_1\rho^3 - A_0I_0\rho.$$

Let $A_3I_3 = 1$, $A_2I_2 = 14$, $A_1I_1 = 49$ and $A_0I_0 = 36$. Then all the A_i are positive and

$$F(\rho) = \rho(\rho^2 - 1)(\rho^2 - 4)(\rho^2 - 9).$$

Thus $F(\rho)$ vanishes at $\rho = 1$, 2, and 3 and changes sign at each of these zeros which proves that (1.5) has at least three periodic solutions for small ϵ.

Since (1.2) is assured by $A_3 > 0$ in (1.9) it remains only to show that $f(x)$ satisfies (1.0) and (1.1). Since the A_i are already fixed and since $A_3 > 0$ and $A_0 > 0$ it follows easily that if C is large enough then $Cx = A_3x^6 - A_2x^4 + A_1x^2 - A_0$ has only one negative root, which is near $x = 0$ and will be designated by $-x_1'$, and only one positive root, which is large and will be designated by x_1. Clearly $f(x)$ now satisfies (1.0) and (1.1) and the example is complete.

With polynomials of higher degree the existence of more periodic solutions can be established. (The argument of Serbin fails in that he does not recognize the restricted range of the monotonicity of ϕ, in Lefschetz's notation, as stated by Lefschetz [1, p. 193] and to which Serbin refers.)

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THE COMPRESSIBLE FLOW CORRESPONDING TO A LINE DOUBLET*

By J. W. CRAGGS (*St. John's College, Cambridge*)

Introduction. The use of the hodograph equations for irrotational compressible flow in two dimensions leads, as noticed by Chaplygin[1] and others, to a method of constructing compressible flow patterns by reference to similar problems in incompressible flow. For certain elementary flows, as for example the flow due to a line source, a line

*Received December 11, 1950.

[1]Chaplygin, *On gaseous jets*, Ann. Sci. Moscow Imperial University, Math.-Phys. section 21, (1904).

vortex or a line doublet, the method is particularly simple. Ringleb[2] has investigated these cases, but his diagrams for the doublet do not illustrate what is perhaps one of the most interesting things about the corresponding compressible flow, the type of limit lines and singularities which occur. In the present paper special attention is paid to these points.

The solution in the hodograph plane. For incompressible flow, the complex potential for a line doublet is given by

$$w = -1/z, \qquad qe^{-i\theta} = dw/dz = 1/z^2,$$

where q is the magnitude and θ the inclination to the real axis of the velocity.

Eliminating z we obtain

$$w = -q^{1/2}e^{-i\theta/2},$$

whence

$$\psi = q^{1/2} \sin \theta/2, \tag{1}$$

where ψ is the stream function. The two branches of (1) correspond to the two half-planes, on each side of the doublet axis. If only the upper half-plane is considered the hodograph plane is once covered, but is cut from the origin along $\theta = 0$ to $q = 1$.

That solution of the hodograph equations of compressible flow which corresponds to (1) is

$$\psi = q^{1/2} f(\tau) \sin \theta/2, \tag{2}$$

where

$$\tau = q^2/Q^2$$

Q being the maximum velocity, corresponding to zero density, and

$$f(\tau) = F(a, b; 3/2; \tau) \tag{3}$$

where F is a hypergeometric function with

$$ab = -3\beta/8, \qquad a + b = (1 - 2\beta)/2,$$

being the reciprocal of $\gamma - 1$, where γ is the adiabatic index.

The hypergeometric series (3) converges rapidly and is easily calculated for values of $\tau < 1$. Near $\tau = 1$ the expansion in terms of $1 - \tau$ may be used.* Thus the flow pattern in the hodograph plane is found.

Reversion to the physical plane. The coordinates in the physical plane, x, y, are given by

$$dx = (d\phi/q) \cos \theta - (\rho_0/\rho)(d\psi/q) \sin \upsilon$$

and

$$dy = (d\phi/q) \sin \theta + (\rho_0/\rho)(d\psi/q) \cos \theta, \tag{4}$$

where

$$q(\partial\phi/\partial q) = -(\rho_0/\rho)(1 - M^2)(\partial\psi/\partial\theta),$$

$$(\partial\phi/\partial\theta) = (\rho_0/\rho)q(\partial\psi/\partial q), \tag{5}$$

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[2]Ringleb, *Exact solutions of the differential equations of an adiabatic gas flow*. Z.A.M.M. (1940).

*The form of the alternative expansion may be found in any standard text on hypergeometric functions.

M being the local Mach number, given by

$$M^2 = 2\beta\tau/(1 - \tau)$$

and (ρ_0/ρ) the density ratio, given by

$$(\rho_0/\rho)^2 = (1 - \tau)^{-2\beta}.$$

Equations (2), (4) and (5) lead to expressions for the derivatives of x and y with respect to q and θ. Integration of those expressions then gives

$$x(\rho q^{1/2}/\rho_0) = 2\tau(df/d\tau)(\cos \tfrac{1}{2}\theta - \tfrac{1}{3}\cos \tfrac{3}{2}\theta) + f\cos \tfrac{1}{2}\theta$$

and (6)

$$y(\rho q^{1/2}/\rho_0) = 2\tau(df/d\tau)(\sin \tfrac{1}{2}\theta - \tfrac{1}{3}\sin \tfrac{3}{2}\theta) + f\sin \tfrac{1}{2}\theta.$$

The flow in the hodograph plane may be found by plotting streamlines $\psi = \text{const}$. The streamlines in the physical plane are then obtained by substituting corresponding values of q and θ in (6).

Singularities of the transformation. Now we investigate the singularities of the transformation. The transformation from the hodograph to the physical plane ceases to be regular when

$$J = \partial(q, \theta)/\partial(x, y)$$

vanishes or becomes infinite.[3] The Jacobian reduces to

$$J = -(\rho/\rho_0)^2 q^3/\{q^2\psi_q^2 + (1 - M)^2\psi_\theta^2\}.$$

Zeros of J can occur only when $q = 0$ or ψ_q or $\psi_\theta \to \infty$. Now f and $df/d\tau$ are always finite so the last two possibilities are eliminated by (2). As $q \to 0$, $f \to 1$ and $J \to 0$; this corresponds to a singular point (the point at infinity) in the flow plane, and a branch point in the hodograph plane. The singularity is due to the symmetry of the flow about the x and y axes, which implies the same velocity vector at (x, y) and $(-x, -y)$. This has already been dealt with by the cut in the hodograph plane.

The singularity corresponding to $J \to \infty$ is of more interest. $J \to \infty$ implies $q^2\psi_q^2 + (1 - M^2)\psi_\theta^2 = 0$. This can only be satisfied for supersonic velocity, $M > 1$. Put $\alpha^2 = M^2 - 1$, then the condition is

$$q\psi_q \pm \alpha\psi_\theta = 0, \tag{7}$$

that is

$$(2\tau f' + \tfrac{1}{2}f)q^{1/2}\sin \tfrac{1}{2}\theta \pm \tfrac{1}{2}\alpha q^{1/2} f\cos \tfrac{1}{2}\theta = 0,$$

whence

$$\alpha \cot \tfrac{1}{2}\theta\,(4\tau f'/f) + 1. \tag{8}$$

The singular line in the hodograph plane, given by (7), is a double loop curve touching the sonic circle, $M = 1$, at $\theta = 0$, crossing itself at $\theta = 180°$, $4\tau(f'/f) + 1 = 0$, ($\tau = .451$, approximately) and touching the limit circle, $q = Q$ at $\theta = 180°$. It is shown in Fig. 1 together with plots of typical streamlines in the hodograph plane.

[3] Cr[illegible]gs, *The breakdown of the hodograph transformation for compressible flow in two dimensions.* Camb. Phil. Trans. **44**, 1948.

The limit line in the flow plane may now be found by substitution in (6) of (8). Corresponding to the double point there are zeros of $dx/d\tau$ and $dy/d\tau$ taken along the

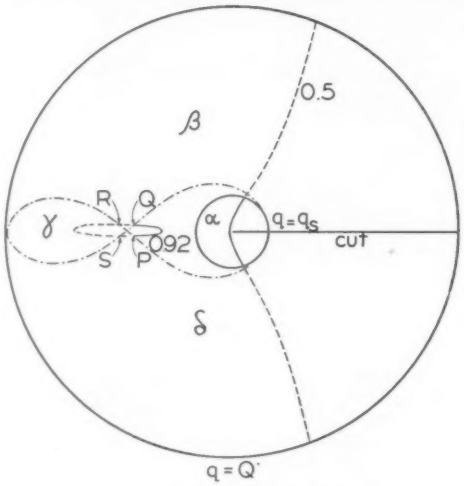

FIG. 1. Hodograph plane.

curve, and each limit line is cusped. It is easily deduced from (2) and (5) that at this point

$$\psi_q = \psi_\theta = \phi_q = \phi_\theta = \psi_{q\theta} = \phi_{qq} = \phi_{\theta\theta} = 0$$

whilst $$q\phi_{q\theta} = (\rho_0/\rho(\alpha^2\psi_{\theta\theta}$$

and $$q^2\psi_{qq} = \alpha^2\psi_{\theta\theta},$$

thus the ratio $d\phi/d\psi$ corresponding to given $dq/d\theta$ at this point is

$$\frac{d\phi}{d\psi} = \frac{2\alpha^2(\rho_0/\rho)q(dq/d\theta)}{\alpha^2(dq/d\theta)^2 + q^2}.$$

But on a limit line $qd\theta/dq = \pm\alpha$, whence $d\phi/d\psi = \pm\alpha(\rho_0/\rho)$. It follows from (4) that

$$\frac{dy}{dx} = \frac{\alpha\cos\theta \mp \sin\theta}{\alpha\sin \pm \cos\theta} = \mp\alpha \qquad \text{at} \qquad \theta = 180°.$$

Next, since $\psi q_\theta = 0$ and ψ_{qq}, $\psi_{\theta\theta}$ are of the same sign, we see that the point is an extremum of ψ and hence that lines $\psi = \text{const.}$ encircle the point and no streamline passes through it. The singularity is of order two (see ref. 3).

The flow in the physical plane. We may now see the main characteristics of the flow in the physical plane. First we draw in the limit lines, shown chain dotted in Fig. 2. Each of the four branches leaving the point B is the join of two sheets of a Riemann surface, the four sheets of which correspond to the regions α, β, γ, δ of Fig. 1. The regions β and δ are separated by the cut $\theta = 0$.

The region α corresponds to a sheet covering the whole of the diagram above the lower pair of limit lines, BP, BQ. Along BQ it joins a sheet (β) bounded by BQ and BR.

The sheet γ extends from BR to BS and the fourth, (δ) bounded by BS and BP, is symmetrical to the second, and joins the first (α) along BP. A streamline, like that for $\psi = 0.92$, which is closed in the hodograph plane, passes over the four sheets in order.

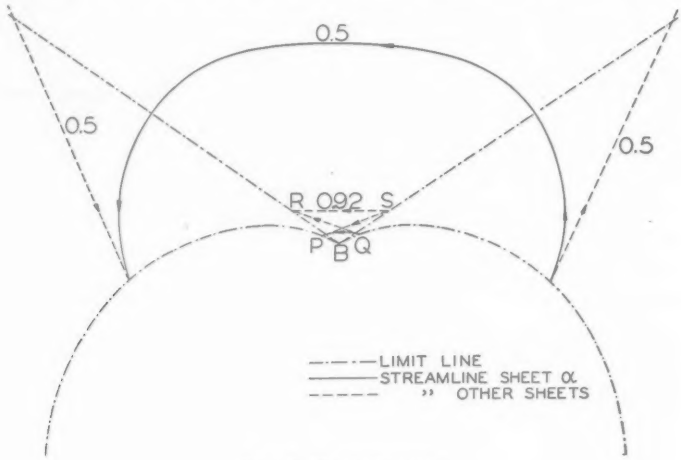

FIG. 2. Physical plane.

The stream line for $\psi = 0.5$, which is also shown, never reaches the sheet γ but goes off, after reflexion at each point where it meets the limit lines, to infinity with speed at infinity of Q.

TABLE I. *The Limit Line.*

τ	θ ° ′	x	y
.16	0.00	1.278	.000
.18	58.36	1.032	.684
.20	87.48	.715	.906
.22	107.00	.555	.986
.24	121.00	.422	1.015
.26	132.12	.312	1.020
.28	141.24	.230	1.014
.30	148.48	.167	1.007
.35	162.42	.062	.997
.40	172.36	.014	.959
.45	179.20	.001	.948
.50	184.14	.010	.955
.55	187.42	.041	.975
.60	190.02	.095	1.003
.65	190.42	.184	1.063
.70	191.10	.320	1.170
.75	192.00	.589	1.314
.80	190.44	.965	1.600
.85	189.56	1.859	2.060
.90	188.02	4.280	3.760
.95	183.44	18.3	15.4
1.00	180.00	$+\infty$	$+\infty$

TABLE II. *Streamlines*

	$\psi = 0.5$		$\psi = 0.9$		$\psi = 0.92$	
τ	x	y	x	y	x	y
.00						
.02	.769	1.618				
.03	.92	1.347				
.04	.962	1.190				
.05	.975	1.087				
.10	.961	.860				
.12	.952	.824				
.14	.945	.801				
.16	.942	.788				
.18	.940	.783				
.20	.940	.784				
.22	.943	.790				
.24	.947	.801	.065	1.017		
.26	.954	.816	.105	1.007		
.28	.962	.836	.123	1.001		
.30	.973	.859	.131	.997	.042	.983
.35	1.009	.938	.124	1.000	.068	.976
.40	1.063	1.051	.096	1.019	.054	.983
.45	1.138	1.207	.049	1.049	.019	.998
.50	1.241	1.424	−.018	1.091	−.031	1.018
.55	1.387	1.731	−.108	1.148	−.904	1.042
.60	1.599	2.181	−.230	1.219	−.163	1.067
.65	1.922	2.822	−.397	1.318	−.234	1.092
.70	2.439	3.956	−.627	1.441	−.126	1.091
.75	3.333	5.986	−.959	1.595		
.80	5.09	9.97	−1.426	1.798		
.85	9.25	19.59	−1.766	2.022		
.90	22.8	51.85				
.95	117.4	282.6				
1.00						

For clarity only $\psi = 0.5$ and $\psi = 0.92$ are shown in the diagrams.

BOOK REVIEWS

Mécanique des milieux continus et déformables. By Maurice Roy. Gauthier-Villars, Paris, 1950. Volume I, xxii + 363 pp., 2800 fr. Volume II, xii + 350 pp., 2300 fr.

M. Roy presents an account of the mechanics of fluids and elastic solids that is based upon the application of thermodynamic principles to continua. The work covers substantially the content of a second year course of lectures at l'École Polytechnique, and the object is to give a synthetic account of the fundamental principles and methods of this mechanics bearing in mind applications to engineering.

Each volume comprises two parts together with extensive appendices; the four parts are entitled "thermodynamics and mechanics of continuous and deformable media", "theory of elasticity", "equilibrium and flow of fluids", and "theory of machines". The principal contents of these parts are described briefly as follows. Part I. "Definitions and fundamentals of deformations"; finite and infinitesimal strains, Eulerian and Lagrangian co-ordinates, acceleration, equation of continuity. "Fundamental thermodynamic principles and equations"; thermodynamic variables, first and second laws of thermodynamics, internal potential, general thermodynamic equation. "General theorems of mechanics"; stability of equilibrium, reciprocal theorems. "Applications to deformable media and elastic solids";

distinction between elastic solids and fluids, stress, infinitesimal deformation of elastic solids, Hooke's law, coefficients of elasticity, equations of equilibrium, isotropic and homogeneous solids, elastic constants. "Application to viscous and non-viscous fluids"; thermodynamic definition of a fluid, viscosity, Navier's hypothesis, dissipation function, equations of continuity and equilibrium and state, Reynolds and Mach numbers. Part II. "Generalities, problems and equations and compatibility"; elastic equilibrium, Clapeyron's formula, Castigliano's theorem, elastic limit. "Plane elasticity"; plane stress and strain, Airy's stress function, use of complex variables, examples. "Three-dimensional elasticity"; simple tension problems, simple flexure of beams, pure torsion, Saint-Venant's problem, beam theory, theorem of three moments. "Thin bodies and elastic bars"; elastic properties of bars, stability. "Small motions of three-dimensional elastic bodies"; dilatational and rotational waves. Vibration of thin bars is considered in the appendix. Part III. "Statics of fluids"; equilibrium conditions, stability of floating bodies. "Dynamics of non-viscous fluids"; Eulerian equations of motion, Helmholtz' circulation theorem, Torricelli's and Saint-Venant's formulae, d'Alembert's paradox. "Plane irrotational flow of incompressible fluids"; velocity potential, stream function, complex potential, singularities in flow field, Kutta-Joukowsky hypothesis, Blasius' formulae, Kutta-Joukowsky theorem, aerofoil theory, images, dead-water regions and cavitation, jets, vortexes, Kármán vortex streets. "Three-dimensional irrotational flow of incompressible fluids"; sources and sinks, images, Rankine solids. "Three-dimensional rotational flow of incompressible fluids"; free and bound vortex sheets, velocity induced by vortex ring, Prandtl's lifting-line theory, air-screw theory. "Wave propagation"; propagation of small disturbances in compressible fluid, shock waves, Hugoniot relations, stationary shock, waves on a canal. "Compressible fluid flow"; potential flow, one-dimensional motion and blast waves, rectilinear singularities in flow field, plane subsonic potential flow with the use of hodograph plane, Prandtl-Meyer expansions, characteristics, simple waves. "Dynamics of viscous fluids"; slow motions, laminar motions, turbulent motions, hydraulics, stability of laminar flow, Prandtl's boundary layer theory, transition, separation. Part IV. "Case of a periodic regime"; fundamental theory, application to turbo-machinery. "Case of a varying regime"; application to hydraulic turbines, stability of regime.

The reviewer commends the author for the clear and stimulating treatment of his subject. The work is a useful contribution to the literature on the mechanics of continua. The reviewer regrets that the tensor notation, with the summation convention, is not used; as Harold Jeffreys has remarked, this notation, although not a necessity, is a great convenience and, when cartesian axes are employed, is not complicated. If discussion were given of this notation, then, in the reviewer's opinion, the presentation of the fundamental analysis would be considerably improved. The list of contents, although very detailed, does not avoid the need for an index. A rather surprising feature of the work is that comparatively few references are given.

H. G. Hopkins

Fourier transforms. By Ian N. Sneddon. First Edition. McGraw-Hill Book Company, Inc., New York, Toronto, London, 1951. xii + 542 pp. $10.00.

The aim (stated in the preface) of this book is to present the theory of Fourier transforms and related topics in a form suitable for the use of students and research workers interested in the boundary value problems of physics and engineering. The book is written for those interested in the application of the theory rather than the theory itself.

The theory of Fourier transforms, of Hankel transforms and of finite Fourier and finite Hankel transforms is covered in the first three chapters (90 pages) of the book. In this section of the book there appears, among other things, discussion of integral transforms in general, Fourier integral theorem, LaPlace transform, Mellin transform and multiple Fourier transforms, Hankel inversion theorem, Hankel transforms of the derivatives of a function, relation between Hankel and Fourier transforms, dual integral equations and the Mellin transform. The Wiener Hopf technique and the delta function are discussed and used in later chapters. The transform theory discussed in the first fifth of the book is applied to physical problems of sufficient difficulty to be of value to the research worker.

The chapter headings in that part of the book devoted to applications are as follows: 4: Theory of Vibrations, 5: The Conduction of Heat in Solids, 6: The Slowing Down of Neutrons in Matter, 7: Hydrodynamic Problems, 8: Applications to Atomic and Nuclear Physics, 9: Two Dimensional Stress Systems, 10: Axially Symmetric Stress Systems, Appendix A; Some Properties of Bessel Functions, Appendix B: Approximate Methods of Calculating Integral Transforms, Appendix C: Tables of Integral Transforms (Fourier, LaPlace, Mellin, Hankel, Finite Fourier, Finite Hankel).

The text has many virtues, one of which is the development of the discussion in each chapter from fundamentals; something that students will certainly appreciate. The book seems to be a very valuable contribution to the literature of physics and applied mathematics.

ROHN TRUELL

Introduction to the study of aircraft vibration and flutter. By R. H. Scanlan and R. Rosenbaum. The Macmillan Company, New York, 1951. x + 428 pp. $7.50.

This text is an introductory treatment of questions of mathematics, structural theory and aerodynamics which are of importance for the analysis of aircraft vibrations, forced or self-induced. It is written for senior or graduate engineering students but should also be useful to others wishing to learn the rudiments of the problem of "flutter".

E. REISSNER

Quantum mechanics of particles and wave fields. By Arthur March. John Wiley & Sons, Inc., New York, and Chapman & Hall, Limited, London, 1951. x + 292 pp. $5.50.

It might seem difficult to justify another book on quantum mechanics, yet this book by Arthur March seems to cover concisely, and well, quite a few topics normally omitted in the elementary texts on quantum mechanics. The chief virtue of the book seems to be that it helps to bridge the gap between elementary texts and the current literature. It will be found that about half of the book deals with relativistic quantum mechanics. The book is concerned primarily with the exposition of general principles and it omits methods of calculation and many applications. The result seems to the reviewer to justify the omissions.

The last chapter deals with the idea of universal length, an idea apparently first set forth by the author. While the ideas of fundamental length do not seem to be a part of current theory, calculations based on these ideas do, in some cases at least, give agreement with experiment.

ROHN TRUELL

Dimensional analysis and theory of models. By Henry L. Langhaar. John Wiley and Sons, Inc., New York, and Chapman & Hall, Limited, London, 1951. xi + 166 pp. $4.00.

Dimensional Analysis and the Theory of Models are subjects which have been in common engineering use for many years. The basic ideas are so elementary that adequate practical results can frequently be obtained without a very rigorous understanding of the basic ideas on which the process rests. In this book, the author attempts to bring together a complete picture of these subjects. The first two chapters in fact, are devoted to a discussion of dimensional units, the elementary development of Buckingham's Theorem and a series of applications to elementary problems. The third chapter is devoted to the systematic procedures by which dimensionless products may be deduced. Thus, the

first three chapters constitute what is sometimes regarded as the whole of dimensional analysis based upon intuitive arguments.

In chapter 4, Algebraic Theory of Dimensional Analysis, the author has brought together a simple but concise treatment of what is now known of the mathematical foundations of dimensional analysis. In this chapter, Buckingham's Theorem is proved without appeal to the possible expansibility of various functions into power series. After thus providing the sound foundation for dimensional analysis, the book takes up in succession a series of specific examples. The items considered include Ship Propeller, Underwater Explosions, Gas Flow, Archery Bow, Bending of Materials, Small Deflection of Elastic Structures, Free Vibrations, Turbulence, Boundary Layers, Run off From a Water Shed, Centrifuga-Compressors, Heat Transfer in a Pipe, Condensation, Transient Heat Transfer, Thermistors, Piezo-electric Receivers, etc. Before taking up this last problem, a careful treatment is given to electrical and magnetic units from the point of view of dimensional analysis.

The treatment of model theory from the point of view of model—prototype similarity starting with the differential equations is considered in chapter 10.

The book makes no pretense to contribute to knowledge on the way which a person must proceed to select the appropriate variables and appropriate basic dimensions for a given problem. To be sure, the numerous examples would assist the student in attaining an understanding of how to go about it, but the basic problem still remains. Thus, no clear understanding is yet available to guide the uninitiated with this task. The reviewer does not consider the problem satisfactorily disposed of by the statement that experience is the only guide. This is, of course, true at the present time but is to be regarded as an unsatisfactory temporary situation.

For anyone uninitiated in the ways of dimensional analysis and the use of models or anyone wishing an excellent systematic and thorough treatment of the subject, this book is excellent.

HOWARD W. EMMONS

Tensor Calculus. By J. L. Synge and A. Schild. University of Toronto Press, Toronto, 1949. xi + 324 pp. $6.00.

This book presents tensor calculus in a manner well suited both to those interested in an abstract study of geometry of spaces with general dimensionality, and to the increasing number of those interested in the applications of tensor calculus to mathematical physics and engineering. This book is an excellent classroom text, since it is clearly written, contains numerous problems and exercises, and at the end of each chapter has a summary of the significant results of the chapter.

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In Chapter 7 there is a return to a general Riemannian space, and such topics as relative tensors, volume, Green's theorem and Stokes' theorem are discussed. Chapter 8 deals with certain non-Riemannian spaces, in which there is no metric tensor and the tensor derivative is defined through the introduction of a linear connection.

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